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Loop amplitudes in maximal supergravity with manifest supersymmetry

Martin Cederwall
Anna Karlsson

Fundamental Physics
Chalmers University of Technology
SE 412 96 Gothenburg, Sweden

Abstract

We present a description for amplitude diagrams in maximal supergravities obtained by dimensional reduction from $D = 11$, derived from a field theory point of view using the pure spinor formalism. The advantages of this approach are the manifest supersymmetry present in the formalism, and the limited number of interaction terms in the action. Furthermore, we investigate the conditions set by this description in order for amplitudes in maximal supergravity to be finite in the ultraviolet limit. Typically, there is an upper limit to the dimension, set by the loop order, which for an arbitrary number of loops is no larger than two. In four dimensions, the non-renormalisation power of the formalism fails for the 7-loop contribution to the 4-point amplitude, all of which is in clear agreement with previous work.

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1 Introduction

In maximal supergravity, one remaining question concerns the behaviour of amplitude diagrams, especially as to the finiteness of the theory. While maximally supersymmetric Yang–Mills theory (SYM) [1] has proven to be perturbatively finite in four dimensions [2–4], a corresponding statement is not available for supergravity — it is still unknown whether or not $\mathcal{N} = 8$ supergravity [5] (the dimensional reduction of $D = 11$ supergravity [6–8]) is perturbatively finite, or where in perturbation theory the first divergences may set in. The general conditions on the formulation in order for it to be finite still have to be shown. The prevalent opinion seems to be that supergravity on its own is not a sensible quantum theory, and that the ultraviolet completion is string theory or M-theory, although there are dissenting views (see e.g. ref. [9]). It is often assumed that the 4-point amplitude has a divergence at 7 loops [10–13].

The techniques for determining the properties above typically make use of maximally supersymmetric quantum field theories, which have been of considerable interest lately. However, the low energy limit of M-theory, maximal supergravity, appears to diverge in the ultraviolet limit for $D \geq 2$ [11, 12, 14], though explicit calculations for the four-graviton amplitude in four dimensions ($\mathcal{N} = 8$) show no divergences up to four loops [15–18], which is as far as detailed examinations have been conducted, so far. This latter procedure is complicated and time consuming, which is why it is interesting to explore alternative descriptions. One such is to use pure spinors as in ref. [12].

The pure spinor formalism has been of use in maximally supersymmetric theories and string theory for some time, its main advantage being the fact that it provides a way of having manifest supersymmetry present. In a maximally supersymmetric model, the ordinary construction in terms of superfields works differently than in less supersymmetric situations, in that supersymmetry transformations close under commutation only modulo equations of motion. This means that the supermultiplets are representations of supersymmetry only on-shell, and there is no way of formulating an action principle with these superfields only, since they do not provide auxiliary fields for off-shell supersymmetry. Pure spinor superfield theory provides a way to encode the traditional superspace constraints, leading to the on-shell multiplets, as equations of motion. This requires the introduction of new bosonic variables, the pure spinors.

It was first observed in ref. [19], that a certain representation of the Lorentz group was needed in order to lift the equations of motion of $D = 10$ SYM, and that this representation could be associated with pure spinors. Other early works [20, 21] noted a connection between supersymmetry and pure spinors, both for gauge theory and supergravity. The breakthrough came with the realisation by Berkovits that the long sought for covariant quantisation of manifestly super-

symmetric superstring (or superparticle) theory could be achieved using a set of variables that indeed are pure spinors [22, 23]. Independently of this advance, in connection to the search for higher-derivative deformations of maximally supersymmetric models [24–28], it was realised that the on-shell property could be understood in terms of cohomology¹ [29]. The ensuing field–antifield structure of maximally supersymmetric field theory has been investigated and actions have been given for virtually all models that do not involve self-dual fields [23, 30–35], including supergravity.

A generic feature of interactions in pure spinor field theory is that the maximal order of the interaction terms is lower than in a component formulation. In supergravity, this even makes the interactions polynomial [34], a phenomenon which has also been observed for the abelian Born–Infeld theory [36]. Having a full interacting field theory with a limited number of interaction terms is a great advantage when it comes to perturbation theory, which we will use in the present paper. It indeed gives a more direct explanation of many of the properties of amplitudes [16, 17, 37–39] that are difficult to discern from a component action. Previous approaches to perturbation theory using pure spinors, e.g. ref. [12], have relied on first quantisation, a construction of a description with pure spinors through comparisons with string theory. In such a formulation, the consistency of every vertex (and eventually the amplitudes) with respect to the symmetries of the theory must be checked, something which the consistency of the classical action takes care of in a field theory. We have been much inspired by the work of ref. [12], and indeed use similar techniques in the concrete calculations. The use of $D = 10$ pure spinors for amplitude calculations in refs. [11, 12, 40–42], with its separation of left- and right-movers is probably the ideal realisation of the Kawai–Lewellen–Tye relations [39], but does not allow for a field theory formulation, in contrast to the $D = 11$ pure spinors used here, which only calls for a single BRST operator.

The paper is organised as follows. In order to apply the pure spinor formalism of $D = 11$ supergravity to the construction of amplitude diagrams, we begin by going through the key concepts of the pure spinor formalism in relation to supergravity in section 2. These are generally well known, but crucial for the subsequent part, section 3, where we address the question of how to describe amplitude diagrams, both tree diagrams and loop diagrams. The main ingredients for the construction of amplitudes are either given by the action, or by quite straightforward generalisations to $D = 11$ pure spinors of techniques known from $D = 10$.

The last part of the amplitude description though, consists of the analysis of the detailed properties of loop amplitudes. These are intricate, but of great use in section 4, the part on the UV behaviour of the amplitude diagrams. Despite

¹Originally termed “spinorial cohomology”, it was soon realised that this is exactly what is obtained from a pure spinor formulation.

the different formalism, our final results are in complete agreement with those of refs. [11, 12]. However, our analysis might not be exhaustive, and might give at hand more properties, not yet noted, despite the intricateness of the description, as is noted in section 5, where our results are summarised and discussed. A few useful spinor identities are listed in appendix A, the zero-mode cohomology of the pure spinor superfield is given in appendix B and appendix C contains a derivation necessary for the standard procedure of general regularisation of amplitude diagrams to be applicable to supergravity.

2 Characteristics of the pure spinor formalism

The advantage of the pure spinor formalism is that it provides an off-shell formulation² for maximally supersymmetric theories. Typically the fields depend not only on the ordinary superspace coordinates, but also on a bosonic spinor λ^α which is pure in the sense that it obeys the constraint $\lambda\gamma_a\lambda = 0$.

The reason for the introduction of the pure spinor λ^α is that the equations of motion for the physical fields of a free, maximally supersymmetric theory tend to follow a pattern which is reproduced by

$$Q = \lambda^\alpha D_\alpha = \lambda D \quad (2.1)$$

where D_α is the covariant fermionic derivative, acting on a pure spinor superfield $\psi = \lambda^{\alpha_1} \dots \lambda^{\alpha_n} C_{\alpha_1 \dots \alpha_n}(x, \theta)$. C here is a superfield of the original theory, dependent only on the superspace coordinates, which contains all the physical fields of the theory. The parts of $Q\psi$ which vanish due to the pure spinor constraint correspond exactly to the terms which are absent in the original equations of motion³. The linearised dynamics of the original (free) theory is captured in full by one single equation of motion of the free enlarged theory: $Q\psi = 0$.

In $D = 11$ supergravity, this construction appears as $\psi = \lambda^\alpha \lambda^\beta \lambda^\gamma C_{\alpha\beta\gamma}(x, \theta)$ [33]. The difference from the original theory is the presence of the pure spinor, which is endowed with ghost number 1 in order to separate physical fields and variables from the original theory (with ghost number zero) from other components of the enlarged theory. To be precise, these contain the set of ghosts and antifields appropriate for the theory, with the correct ghost numbers. The general field $\psi(x, \theta, \lambda)$ (the only field present) is fermionic, with ghost number 3 and no

²That is, a formulation which can describe fields that do not obey the equations of motion, and can provide an action principle.

³From the point of view of a superspace gauge theory, this happens due to conventional constraints (see e.g. ref. [43] for a full discussion). These are exactly what is needed in order to describe a theory in terms of the lowest-dimensional superfield. In the present case $C_{\alpha\beta\gamma}$ is the part of the superspace 3-form with only fermionic indices.

free indices, and has as its λ - and θ -independent part the ghost for ghost for ghost of the tensor gauge symmetry. It includes, in addition to the unconstrained superfield $C_{\alpha\beta\gamma}$ of ghost number zero, also superfields containing the relevant ghosts and antifields of the theory (and more).

In short, Q can be recognised as a nilpotent operator, and the cohomology of Q represents the free on-shell fields, including ghosts and antifields. The actual field content can be read off in table 1 in appendix B. There, the Lorentz representations of the zero-mode cohomology of Q is given. The proper cohomology of Q contains these fields, constrained by some differential equation (equation of motion) which also is in the zero-mode cohomology. Through a separation of terms of ghost number zero from the rest, the enlarged theory gives at hand results in the original theory. At the same time, the presence of Q enables off-shell calculations. The details of this — the main concepts of the formalism — is what we will begin by describing, below, before dealing with the details of the formalism.

In the end, the full action of $D = 11$ supergravity contains more than the kinetic operator Q . In addition to the term of the free theory, describing the free propagation of a field, it contains a 3-point and a 4-point coupling⁴: [34]

$$S = \frac{1}{\kappa^2} \int [dZ] \left(\frac{1}{2} \psi Q \psi + \frac{1}{6} (\lambda \gamma_{ab} \lambda) \left(1 - \frac{3}{2} T \psi \right) \psi R^a \psi R^b \psi \right) \quad (2.2)$$

The precise meaning of the operators involved, as well as the integration measure, will be elucidated later in this section. Importantly, the pure spinor formalism transforms the supergravity action into a polynomial expression. The quartic term is the last term and moreover, the operator T is nilpotent so that a 4-point vertex at most can appear once in a diagram. However, though the pure spinor formalism simplifies the interaction terms in such a way, some parts remain complicated. For example, it is not known how to write down a solution for ψ . Although the formulation enjoys the full gauge symmetry of $D = 11$ supergravity (diffeomorphisms, local symmetry and tensor gauge symmetry), the geometrical picture is obscured, as is the background invariance. In a way, the complications have only been transferred between different parts of the theory. For perturbative theory though, the action above is ideal.

The parts of this action is what will be described below, after the main concepts have been presented. To begin with, we discuss the formalism with respect to what variables there are, including a discussion on what the pure spinor really is: the solution to the pure spinor constraint. Notably, the action is expressed in the so called non-minimal formalism, which is what is usually used for the pure spinor

⁴Note that the overall constant of the action, here displayed as the coupling constant κ^{-2} , depends on the normalisation of ψ . For canonically normalised fields, it will appear in the interaction terms instead.

formalism, since the introduction of two more sets of variables (apart from the pure spinor mentioned above) enables integration. The introduction of the concept of integration, and the ensuing clarification of the importance of the non-minimal formalism, is what is described right after the non-minimal superspace.

The final part of the formalism we present, before heading off for new theoretical aspects concerning the construction of amplitude diagrams, is the operators involved in the action. It should be noted that (most) concepts in this section are well known, though the treatment of the solution to the pure spinor constraint in this formalism, see subsection 2.2.1, only has been presented to linearised order in Ω before [44]. Here, it will be treated in full.

2.1 The heart of the formalism

The identification of the Q described above is performed in a very ad hoc manner. However, the theory it gives rise to is very well defined. As the covariant derivative D_α obeys $\{D_\alpha, D_\beta\} = -2\gamma_{\alpha\beta}^a \partial_a$ in flat space and similar relations for the superspace torsion and curvature, annihilated by the pure spinor condition, hold for other backgrounds⁵, Q can be recognised as a nilpotent operator, a BRST operator. This operator is not a supersymmetry transformation operator, but commutes with all flat superspace Killing spinors so that the global supersymmetry transformations close both on- and off-shell. Essentially, the action is recognised to give at hand a theory with the correct properties, so that it represents a working action for the free theory. It has to be remembered, though, that ghosts and antifields are included, and need to be treated properly by gauge fixing.

The BRST construction concerns free theories, but it has a natural extension to a theory of interactions, a field theory formulation: the Batalin-Vilkovisky (BV) formalism [45,46]. There, the BRST symmetry is central, but while the BRST formulation can be compared to a first quantised theory, the BV represents a second quantised version (a field theory), a change which usually requires the introduction of new fields (ghosts and antifields). The alteration corresponds to exchanging the BRST symmetry, where Q gives rise to the variations, for the BV symmetry, where the generalised action gives rise to the variations of the fields: $(S, \psi) = \delta\psi$.

This antibracket usually is expressed in the introduced fields and antifields. However, all those fields are already present in the field ψ , which is its own antifield, so for the pure spinor formalism, the extension to a BV formalism is performed through another ad hoc recognition of how to imitate the BV formalism.

⁵In this paper, the background will be assumed to be flat, since we are aiming at Minkowski space amplitudes. The main difference with another background present would be that the other operators of the theory would need to be constructed from covariant derivatives while taking the background into consideration. Such a procedure would be a bit more intricate, though probably entirely feasible. [34]

The only antibracket possible to write down, where the field ψ is self-conjugate with respect to the antibracket, is:

$$(A, B) \sim \int \frac{\delta A}{\delta \psi} \frac{\delta B}{\delta \psi} [dZ] \quad (2.3)$$

This simple construction turns out to be the correct one, which can be checked on the component fields appearing in the cohomology listed in appendix B.

This concludes the recognition of what the pure spinor formalism really is. The expression for the equation of motion for a field ψ is:

$$(S, \psi) = 0 \quad (2.4)$$

If a pure spinor superfield ψ fulfils this equation, and thus obeys the equation of motion, it is termed to be on-shell. Gauge invariance for a field is represented by the exact same expression, as is typical in a BV formalism. Furthermore, the master equation which the action must obey is

$$(S, S) = 0 \quad (2.5)$$

which also gives at hand the form of the action once the 3-point coupling has been deduced from certain properties of the theory, a process which will be described below.

Most importantly, the BV formalism makes it possible to perform off-shell calculations. It incorporates the alterations that are necessary in order to bring the original on-shell theory off-shell, where interactions can be taken into account in a manifestly supersymmetric formalism. The BV formalism incorporates far more than the BRST version through the change of the symmetry transformations. There are many off-shell field components that take part in the interactions, at the same time as the free physical fields can be expressed in the minimal formalism and interpreted with respect to the original theory. Furthermore, gauge symmetry is manifest throughout the process, just as for the SYM theory, so at the point when the gauge needs to be fixed, no extra procedure is necessary in order to check gauge invariance, which is always present. These are considerable strengths of the formalism.

2.2 Superspace in the non-minimal formalism

In $D = 11$, the coordinates which describe superspace consist of 11 bosonic variables (x^a) and 32 fermionic spinors (θ_α). They are symplectic, so $(\lambda\theta) = -(\theta\lambda)$, as they are connected by the asymmetric tensor⁶ $\varepsilon^{\alpha\beta}$. Furthermore, the dimension

⁶For more details, e.g. the general Fierz identity of four components: $(AB)(CD)$, have a look at appendix A.

of a variable x is usually set to -1 , so that the corresponding for a spinor coordinate is $-1/2$. In addition, in the pure spinor formalism there is the pure spinor λ_α , which is bosonic and has 9 degrees of freedom less than ordinary spinors due to the pure spinor condition, i.e., 23 independent ones⁷. In the non-minimal formalism there are also two other variables, introduced due to their properties: $\bar{\lambda}_\alpha$ and r_α . The first of these is bosonic with ghost number -1 , whereas the second is fermionic with ghost number zero, and they obey $(\bar{\lambda}\gamma^a\bar{\lambda}) = (\bar{\lambda}\gamma^a r) = 0$. Both are of dimension $1/2$, so that the set $(r, \bar{\lambda})$ has quantum numbers conjugate to those of (θ, λ) . Note that at the introduction of the non-minimal variables, the BRST operator has to be modified in order for the cohomology not to be changed, see subsection 2.3.1.

Before moving on to the finer points of the theory, a brief explanation of notation is convenient as well. To begin with, the derivative of λ^α is denoted by ω_α . The corresponding for $\bar{\lambda}_\alpha$ is $\bar{\omega}^\alpha$, which illustrates an analogy with barred and unbarred quantities which is used throughout this paper. The derivative with respect to r_α on the other hand, is denoted by s^α . Moreover, all expressions must be gauge invariant, which is why all these derivatives invariably show up in the combinations of [47]:

$$\begin{aligned} N &= \lambda\omega & \bar{N} &= \bar{\lambda}\bar{\omega} + rs & S &= \bar{\lambda}s \\ N_{ab} &= \lambda\gamma_{ab}\omega & \bar{N}_{ab} &= \bar{\lambda}\gamma_{ab}\bar{\omega} + r\gamma_{ab}s & S_{ab} &= \bar{\lambda}\gamma_{ab}s \end{aligned} \quad (2.6)$$

Further notation consists of the two scalars possible to form from the pure spinors in the non-minimal formalism [33]:

$$\xi = (\lambda\bar{\lambda}) \quad \eta = (\lambda\gamma^{ab}\lambda)(\bar{\lambda}\gamma_{ab}\bar{\lambda}) \quad (2.7)$$

The last is not to confuse with η^{ab} , which simply lowers or raises indices.

2.2.1 The solution to the pure spinor constraint

To wrap up the properties of $D = 11$ superspace and the pure spinor constraint, we now turn to the solution of that same constraint. The procedure for finding out that the pure spinor has 23 degrees of freedom goes as follows.

A spinor in $D = 11$ decomposes as $\mathbf{16} \oplus \overline{\mathbf{16}}$ when $so(11) \rightarrow so(10)$, and thus becomes

$$\mathbf{32} \rightarrow \mathbf{1}_{-5/2} \oplus \mathbf{5}_{-3/2} \oplus \mathbf{10}_{-1/2} \oplus \overline{\mathbf{10}}_{1/2} \oplus \overline{\mathbf{5}}_{3/2} \oplus \mathbf{1}_{5/2} \quad (2.8)$$

⁷We stick to the terminology “pure spinor” for the algebraically constrained object λ with $\lambda\gamma^a\lambda = 0$, although it clearly differs from the traditional mathematical definition, which only makes sense in an even number of dimensions.

under the further decomposition $so(10) \rightarrow su(5) \oplus u(1)$. Such a spinor contains all forms:

$$\lambda = \bigoplus_{p=0}^5 \lambda_p \quad (2.9)$$

Some of these will be determined entirely by the pure spinor condition, a condition which however looks a bit different in this notation. The 10-dimensional γ -matrices act as⁸ $\omega \wedge$ and ι_v and in addition to those, there is the $\gamma^{11} = (-1)^p$, p being the form degree. In total, the $so(11)$ -invariant symplectic spinor scalar product is:

$$\langle A, B \rangle = \star \sum_{p=0}^5 (-1)^{\frac{p(p+1)}{2}} A_p \wedge B_{5-p} \quad (2.10)$$

With this, the pure spinor constraint translates into:

$$\langle \lambda, (-1)^p \lambda \rangle = \langle \lambda, \omega \wedge \lambda \rangle = \langle \lambda, \iota_v \lambda \rangle = 0 \quad (2.11)$$

As λ_0 is just a scalar factor and the wedge product is symmetric for λ_{even} , whereas the opposite is true in between λ_{odd} 's, the first and second equation can be used to solve for λ_5 and λ_4 respectively, in terms of the other λ_n 's:

$$\begin{aligned} \lambda_4 &= \lambda_0^{-1} (-\lambda_1 \wedge \lambda_3 + \frac{1}{2} \lambda_2 \wedge \lambda_2) \\ \lambda_5 &= \lambda_0^{-1} \lambda_2 \wedge \lambda_3 - \frac{1}{2} \lambda_0^{-2} \lambda_1 \wedge \lambda_2 \wedge \lambda_2 \end{aligned} \quad (2.12)$$

The last condition on the other hand, the third of the equations in eq. (2.11), corresponds to:

$$\begin{aligned} -2\iota_v \lambda_1 \wedge \lambda_5 + 2\iota_v \lambda_2 \wedge \lambda_4 - \iota_v \lambda_3 \wedge \lambda_3 &= 0 \\ \stackrel{(2.12)}{\Longleftrightarrow} \iota_v (\lambda_3 - \lambda_0^{-1} \lambda_1 \wedge \lambda_2) \wedge (\lambda_3 - \lambda_0^{-1} \lambda_1 \wedge \lambda_2) &= 0 \end{aligned} \quad (2.13)$$

Here, we have rearranged the expression in order to make the solution visibly clearer. Its solution contains a 3-form Ω :

$$\lambda_3 = \lambda_0^{-1} \lambda_1 \wedge \lambda_2 + \Omega, \quad \iota_v \Omega \wedge \Omega = 0 \quad (2.14)$$

This means that the pure spinor is described in full by λ_0 , λ_1 , λ_2 and Ω , where the condition on Ω needs to be examined further.

The solutions for Ω describe, modulo a scale, the Grassmannian $Gr(2, 5) = SU(5)/S(U(3) \times U(2))$ of 2-planes in \mathbb{C}^5 . The 16-parameter solution with $\Omega = 0$ is the space $SO(12)/SU(6) \times \mathbb{R}_+$ of 12-dimensional pure spinors, and the full solution is a bundle over that space with fibre $Gr(2, 5) \times \mathbb{C}$. For a parametrisation

⁸For practical purposes, ω is a 1-form and ι_v represents the contraction with the vector field v , reducing the form degree of the form it acts on by one.

of the solutions, we break $su(5) \rightarrow su(3) \oplus su(2) \oplus u(1)$, so that Ω splits into $\Omega_{(3,0)}$, $\Omega_{(2,1)}$ and $\Omega_{(1,2)}$, where the form degrees are with respect to the $su(3)$ and $su(2)$ parts, respectively. Their charges under the “new” $u(1)$ are -1 , $-\frac{1}{6}$ and $\frac{2}{3}$, set by the assignment of the charge $\frac{1}{3}$ to an $su(3)$ vector and the charge $-\frac{1}{2}$ to an $su(2)$ vector. Furthermore, it turns out to be convenient to dualise the $su(3)$ form indices to vector indices:

$$\star_3 \Omega = \omega + \Omega^1_1 + \Omega^2_2 \quad (2.15)$$

In order to use this though, we need to relate wedge products of 1-forms with forms to contractions of 1-forms with the dual multi-vector, and correspondingly for contractions with vectors. For a general dimension n and form degree p we have

$$\begin{aligned} \star(\omega \wedge A_p) &= (-1)^{n-p+1} \iota_\omega \star A_p \\ \star(\iota_v A) &= (-1)^{n-p} v \wedge \star A_p \end{aligned} \quad (2.16)$$

where wedge products between vectors are defined just as between forms. We also need the general relation between scalars:

$$\star(A_p \wedge B_{n-p}) = \star(\star A_p \wedge \star B_{n-p}) \quad (2.17)$$

Now, the constraint in eq. (2.14) splits into two sets of equations, since the vector v decomposes in $v^{(1,0)}$ and $v^{(0,1)}$. The one containing $v^{(1,0)}$ is:

$$2\omega\Omega^2_2 - \Omega^1_1 \wedge \Omega^1_1 = 0 \quad \Rightarrow \quad \Omega^2_2 = \frac{1}{2}\omega^{-1}\Omega^1_1 \wedge \Omega^1_1 \quad (2.18)$$

The solution happens to be just what it takes for the equation containing $v^{(0,1)}$ to be automatically satisfied. In total, this gives at hand that the complete solution of the $D = 11$ pure spinor constraint is parameterised by the 23 variables $(\lambda_0, \lambda_1, \lambda_2; \omega, \Omega^1_1)$.

We now turn to the expression for the volume form for the pure spinor. It is expected to be given by $[d\lambda] = \lambda_0^\alpha d\lambda_0 d^5\lambda_1 d^{10}\lambda_2 \omega^\beta d\omega d^6\Omega^1_1$ for some numbers α and β , but a counting of the two $u(1)$ charges immediately gives at hand that $\alpha = -5$ and $\beta = -2$. What remains is to show the $so(11)$ invariance of the volume form. The first step in this is to show the $su(5)$ invariance of $\omega^{-2}d\omega d^6\Omega^1_1$, after which one can proceed to the full volume form. The calculations are a bit tedious, but everything works out. The volume form is thus:

$$[d\lambda] = \lambda_0^{-5} d\lambda_0 d^5\lambda_1 d^{10}\lambda_2 \omega^{-2} d\omega d^6\Omega^1_1 \quad (2.19)$$

Here, the seven inverse powers of the pure spinor reflects the highest (ghost anti-field) cohomology at λ^7 . The two powers of ω agrees with the concrete form of

the ghost antifield cohomology in refs. [33, 34], carrying exactly two factors of $(\lambda\gamma^{(2)}\lambda)$.

As a representative pure spinor λ in the generic, 22-dimensional orbit under $Spin(11)$, i.e., one with $(\lambda\gamma^{ab}\lambda) \neq 0$, one can take the pure spinor with only the two numbers λ_0 and ω non-vanishing, and $\lambda_1 = \lambda_2 = \Omega_{(2,1)} = 0$, implying $\Omega_{(1,2)} = \lambda_4 = \lambda_5 = 0$. For this representative, $(\lambda\gamma_{ab}\lambda)$ only has a single non-vanishing component, when the indices are upper $su(2)$ indices, and $(\lambda\gamma^{a'b'}\lambda) \sim \lambda_0\omega\varepsilon^{a'b'}$. The correspondence with the scalar quantities defined in eq. (2.7) is $\xi \sim |\lambda_0|^2$, $\eta \sim |\lambda_0|^2|\omega|^2$. A representative with $\omega = 0$ is in the 15-dimensional orbit of $D = 12$ pure spinors.

The space of pure spinors can be equipped with a Calabi–Yau structure, where the holomorphic top form is precisely the one given in eq. (2.19). This is in analogy with the ten-dimensional case [48], where more details about the metric and Kähler structures were given. The integration described in the following subsection will have a natural interpretation in terms of the Calabi–Yau geometry on pure spinor space.

2.2.2 Integration and divergences

As stated above, the volume form for the pure spinor contains seven inverse powers of the pure spinor, so that the combined volume form for the superspace variables is:

$$[dz] = \lambda^{-7} d^{11}x d^{32}\theta d^{23}\lambda \quad (2.20)$$

However, this does not suffice as a measure factor for the action, as it has the wrong ghost number and dimensional properties. A proper $[dZ]$ in eq. (2.2) must have ghost number -7 and dimension -3 , whereas $[dz]$ has ghost number 16 and dimension -3 . Furthermore, the addition of a function of the superspace variables to $[dz]$ would not do, since such a measure would be degenerate. It would exclude a lot of states in ψ , due to ψ being an expansion in the variables, some parts of which would have higher powers of the superspace variables than that type of measure could take. Neither is it a priori clear how to perform integration over the 23 holomorphic variables λ . A treatment using only these variables typically has to be either non-manifestly covariant or involve picture changing operators [49, 50].

The trick is instead to introduce the non-minimal variables [47] $(\bar{\lambda}_\alpha, r_\alpha)$ and extend the formalism into the non-minimal formalism, which is a possibility due to the properties of Q , which will be discussed below. The previously mentioned conditions on the non-minimal variables: $\bar{\lambda}_\alpha$ a pure spinor with ghost number -1 and r_α a fermionic spinor of ghost number 0 with $(\bar{\lambda}\gamma^a r) = 0$, both of the same dimension $(1/2)$, is just what it takes in order to form a proper measure factor for

the action. In fact, $[d\bar{\lambda}][dr] \sim d^{23}\bar{\lambda} d^{23}r$ since [33, 51, 52]:

$$\begin{aligned} [d\lambda]\lambda^{\alpha_1} \dots \lambda^{\alpha_7} &= \star \bar{T}^{\alpha_1 \dots \alpha_7}_{\beta_1 \dots \beta_{23}} d\lambda^{\beta_1} \dots d\lambda^{\beta_{23}} \\ [d\bar{\lambda}]\bar{\lambda}_{\alpha_1} \dots \bar{\lambda}_{\alpha_7} &= \star T_{\alpha_1 \dots \alpha_7}^{\beta_1 \dots \beta_{23}} d\bar{\lambda}_{\beta_1} \dots d\bar{\lambda}_{\beta_{23}} \\ [dr] &= \bar{\lambda}_{\alpha_1} \dots \bar{\lambda}_{\alpha_7} \star \bar{T}^{\alpha_1 \dots \alpha_7}_{\beta_1 \dots \beta_{23}} \frac{\partial}{\partial r_{\beta_1}} \dots \frac{\partial}{\partial r_{\beta_{23}}} \end{aligned} \quad (2.21)$$

The right properties for $[dZ]$ are thus given by:

$$[dZ] = d^{11}x d^{32}\theta [d\lambda][d\bar{\lambda}][dr] \quad (2.22)$$

The precise tensor structure of this measure is implied by the statement that the tensor⁹ T is obtained as the Clebsch–Gordan coefficients for the formation of a singlet from 7 symmetrised indices and 9 antisymmetrised, both groups of indices being in the irreducible module (020003) [33]:

$$\begin{aligned} T_{\alpha_1 \dots \alpha_7}^{\beta_1 \dots \beta_9} \lambda^{\alpha_1} \dots \lambda^{\alpha_7} \theta_{\beta_1} \dots \theta_{\beta_9} \\ \sim (\lambda\gamma^{ab}\lambda)(\lambda\gamma^{cd}\lambda)(\Lambda^{ijklm}\gamma^n\theta)(\theta\gamma_{abp}\theta)(\theta\gamma_{cdp}\theta)(\theta\gamma_{ijm}\theta)(\theta\gamma_{klm}\theta) \end{aligned} \quad (2.23)$$

Here, Λ_{α}^{ijklm} is in (00003): $\Lambda_{\alpha}^{ijklm} = \lambda_{\alpha}(\lambda\gamma^{ijklm}\lambda) - 2(\gamma^{[ijk}\lambda)_{\alpha}(\lambda\gamma^{lm]}\lambda)$. Alternatively, we have [51, 52]:

$$\begin{aligned} T_{\alpha_1 \dots \alpha_7}^{\beta_1 \dots \beta_9} \lambda^{\alpha_1} \dots \lambda^{\alpha_7} \theta_{\beta_1} \dots \theta_{\beta_9} \\ \sim \epsilon_{a_1 \dots a_{11}} (\lambda\gamma^{a_1}\theta) \dots (\lambda\gamma^{a_7}\theta) (\theta\gamma^{a_8 a_9 a_{10} a_{11}}\theta) \end{aligned} \quad (2.24)$$

Moreover, it is worth mentioning that this full integration over non-minimal pure spinor space has a very natural and elegant geometric formulation [40, 48]. In the “Dolbeault picture”, where the variable r is identified with the antiholomorphic one-form $d\bar{\lambda}$, any function of λ , $\bar{\lambda}$ and r becomes a cochain with antiholomorphic form indices. The integration then is defined using the holomorphic top form Ω of eq. (2.19), and is given as:

$$\int [dZ]\psi = \int \Omega \wedge \psi \quad (2.25)$$

Importantly, the measure factor has the property that it extracts λ^7 for the integration over the pure spinors, part of which is two factors of $\lambda\gamma^{(2)}\lambda$ due to the form of $\star\bar{T}$ given in eq. (2.23) [33]. Furthermore, for a non-zero result the fermionic integrals must be saturated, which means that the integrand must contribute with 32 θ ’s and 23 r ’s.

⁹The \star in eq. (2.21) implies dualisation from 9 to 23 antisymmetric spinor indices.

The latter represents a problem, as there are demands on the integrand in order for the integral not to diverge as well. In particular, there ought not be too many negative factors of the scalars in eq. (2.7), which would represent a divergence at the origin with respect to $(\lambda, \bar{\lambda})$ and/or at the submanifold of complex codimension 7 where η vanishes. It is however convenient to separate those two behaviours as well as to treat factors of $\lambda\gamma^{(2)}\lambda$, which is why the latter divergence is described with respect to σ : $\sigma^2 \sim \eta/\xi^2$. Each factor of $(\lambda\gamma^{ab}\lambda)$ and $(\bar{\lambda}\gamma^{ab}\bar{\lambda})$ goes like σ , and since the submanifold has real codimension 14 and the measure removes two powers of σ , we arrive at $[dZ] \sim \sigma^{11}d\sigma$. In total, the volume form of the pure spinor thus sets the following conditions on the components of an integrand in order for it to be finite: [33]

$$\begin{cases} \xi^x, & x > -23 \\ \sigma^y & y > -12 \end{cases} \quad (2.26)$$

Here, the first condition applies to an integrand $\lambda^7\xi^x$. The requirements are a bit worrisome, since the operators of section 2.3 typically contain negative powers of η . When there is a lot of them present, the integral may give as nonsensical a result as $0 \times \infty$, typically diverging the most severely with respect to σ . On top of that, the part giving at hand a divergence can just as well come from a too high power of $(\lambda, \bar{\lambda})$ as well, in the limit where ξ approach infinity. Typically this is a sign of that some part of the theory is missing, that is, the regularisations for these types of divergences.

However, those properties are here considered to be more tightly connected to the amplitude diagrams than the formalism concerning the action, and will therefore not be presented until the next section. Before any of that though, the operators of the theory need to be presented, as the important building blocks they are.

2.3 The operators of the action

At the heart of the formalism, as mentioned above, is the BRST operator Q which is identified in an ad hoc manner, yet gives at hand a clearly defined and elegant BV formalism. It shapes the rest of the operators, partly in what ways gauge fixing might be done, which gives at hand a condition for the propagator, and most definitely in the expressions for the vertices, which are described by the vertex operators. Most of them turn out to be singular at $\eta = 0$.

2.3.1 The kinetic operator Q and regulators

The BRST operator Q , mentioned in the beginning of this section, changes by an addition of the operator $\bar{\partial}$ with the enlargement of the theory into the non-minimal

formalism:

$$Q = \lambda D + r\bar{\omega} \equiv q + \bar{\partial} \quad (2.27)$$

The notation $\bar{\partial}$ is due to the geometric identification of the variable r with the anti-holomorphic differential $d\bar{\lambda}$ (the “Dolbeault picture”) [40]. This modified Q has the same cohomology as the initial Q , in the minimal formalism, and the proper ghost number and dimensional properties, which is why the alteration is correct. Formally, the first statement is that the operator above has a cohomology which in each class has a representative independent of the non-minimal variables [47, 53], so it incorporates the non-minimal formalism while conserving the desired properties of the minimal formalism. The properties of the original theory can still be extracted.

Q is the kinetic operator of the formalism, and there is one very useful twist to it. The free cohomology of an expression¹⁰ is unchanged at the introduction of a term which is Q -exact, provided that all amplitude calculations are performed between on-shell external states. This is generally assumed, which is why there is a freedom of the formalism: $1 \rightarrow 1 + \{Q, \chi\}$. Typically, this shows up in the use of so called regulators, which can be added or removed at any time:

$$e^{\{Q, \chi\}} = 1 + \{Q, \chi\} + \dots \quad (2.28)$$

The only conditions on χ are such that the exponent makes sense, e.g. with respect to dimension and ghost number, which is why χ must have ghost number -1 . It may even introduce new variables (and corresponding integration measures), in a way similar to the introduction of the non-minimal variables. Consequently, Q would change correspondingly as well. As the terminology hints at, regulators will be of great importance in the regularisations to be described below. Due to the Q -invariance of the theory, one can choose to treat the first term in the expansion of eq. (2.28) or the entire series in order to find the properties of the expression which the regulator is part of. A third possibility is to use the first term with the addition of the term in the series which allows for constructive interpretation of the results. This will come in handy later on.

2.3.2 The propagator and the b -ghost

A typical feature of pure spinor field theories is that gauge fixing cannot be performed in a way similar to what is done in classical field theories. In a BV formalism, this typically requires the identification of separate fields and antifields, some of which need to be introduced in the non-minimal formalism. The process

¹⁰The free cohomology of X is the parts of X that obey $[Q, X] = 0$, with the equivalence relation $X \approx X + [Q, Y]$, where the brackets are set by the statistics of X and Y . Compare with eq. (2.4), which describes the cohomology of ψ .

of gauge fixing involves the elimination of the antifields, which with the use of a gauge fixing fermion become expressed in terms of the fields. The superfield ψ , on the other hand, already contains all such fields¹¹, including the ones necessary in order to fix the gauge, but it is not possible to separate those fields from the rest, without separating ψ into components of definite ghost number. Instead, the principle is to imitate string theory in a covariant gauge known as the Siegel gauge [54], as for the scalar particle. This gives at hand a free propagator b/p^2 :

$$\{Q, b\} = \partial^2 \quad b\psi_{\text{on-shell}} = 0 \quad (2.29)$$

Here, the equation to the right represents the choice of gauge, and the one to the left is the correct behaviour of the b -ghost, the solution of which by default ought to give at hand the property $\{b, b\} = 0$.

Now, the b -ghost is not present as a fundamental variable in the theory to begin with since the BRST operator was not constructed from a symmetry point of view. A free field theory is ordinarily based on, or related to, a first-quantised particle with a world-line reparametrisation invariance. The constraint $p^2 = 0$ (or, in the string theory case, the Virasoro constraint) then is the generator of these reparametrisations, and in effect, there is a corresponding bc reparametrisation ghost system. In contrast, in maximally supersymmetric theories, the equations of motion follow from the condition $Q\psi = 0$, and do not have to be imposed separately. In the absence of reparametrisations, there is no b -ghost. However, the first condition in eq. (2.29) makes it possible to construct a composite b operator with the correct properties. This is not surprising, given the knowledge that the free fields are massless, and there should exist a gauge choice where $p^2 = 0$ for the linearised theory. The construction is known from maximal SYM [41] and superstring theory [41, 47]. It is, in short, the b operator which appears in the propagator in the Siegel gauge.

The solution to b is found from the ansatz $\{\lambda D, b_0\} = \partial^2$ and through an iteration of $\{r\bar{\omega}, b_n\} + \{\lambda D, b_{n+1}\} = 0$, where the lowered indices denote the numbers of r involved in each part of the b operator. It is most easily expressed using terms that are not manifestly gauge invariant, i.e., that contain derivatives ω not in the combinations (2.6), although it can be expressed entirely in the gauge invariant terms of eq. (2.6) as well, which means that it is a well defined part of the manifestly supersymmetric formalism. Note though, that such terms may seem to be $\bar{\partial}$ -exact despite them not being so. A part $[\bar{\partial}, \rho]$ with $\delta^a \rho = [(\lambda \gamma^a \lambda), \rho] \neq 0$, for example the term b_3 in the b operator shown below, cannot be shifted away

¹¹Recall that ψ is its own antifield, as is implicit in eq. (2.3).

through the use of Q -exact terms, as ρ is not a well defined operator¹².

$$\begin{aligned}
b = & \frac{1}{2}\eta^{-1}(\bar{\lambda}\gamma_{ab}\bar{\lambda})(\lambda\gamma^{ab}\gamma^i D)\partial_i \\
& - \frac{1}{12}\eta^{-2}L_{ab,cd}^{(1)}\left[(\lambda\gamma^{abcij}\lambda)\eta^{dk} + \frac{2}{7}(\lambda\gamma^{acijk}\lambda)\eta^{bd}\right]\left[(D\gamma_{ijk}D) - 24N_{[ij}\partial_{k]}\right] \\
& + \frac{1}{2}\eta^{-3}L_{ab,cd,ef}^{(2)}\left[(\lambda\gamma^{ab}\lambda)(\lambda\gamma^{cde}{}_{ij}\lambda)(\omega\gamma^{fij}D) \right. \\
& \quad \left. + \eta^{df}(\lambda\gamma^{ab}\lambda)\left(\frac{2}{7}(\lambda\gamma^{ce}{}_{ijk}\lambda)(\omega\gamma^{ijk}D) + \frac{9}{7}(\lambda\gamma^c{}_i\lambda)(\omega\gamma^{ei}D)\right) \right. \\
& \quad \left. + \frac{1}{3}(\lambda\gamma^{ce}\lambda)\left((\lambda\gamma^{fa}{}_{ijk}\lambda)(\omega\gamma^{dbijk}D) - 9\eta^{df}(\lambda\gamma^a{}_i\lambda)(\omega\gamma^{bi}D)\right)\right] \\
& + \frac{2}{3}\eta^{-4}L_{ab,cd,ef,gh}^{(3)}(\lambda\gamma^{ab}\lambda)(\lambda\gamma^{ce}\lambda)\left[(\lambda\gamma^{dg}{}_{ijk}\lambda)(\omega\gamma^{fhijk}\omega) \right. \\
& \quad \left. - 9\eta^{df}(\lambda\gamma^g{}_i\lambda)(\omega\gamma^{hi}\omega)\right] \tag{2.30}
\end{aligned}$$

This operator was essentially constructed in ref. [57]. For convenience, we have introduced a special short notation used for the dependence on the non-minimal variables, where $\llbracket \dots \rrbracket$ denotes an antisymmetrisation between the $p+1$ antisymmetric pairs of indices:

$$L_{a_0b_0,a_1b_1,\dots,a_pb_p}^{(p)} = (\bar{\lambda}\gamma_{\llbracket a_0b_0}\bar{\lambda})(\bar{\lambda}\gamma_{a_1b_1}r) \dots (\bar{\lambda}\gamma_{a_pb_p}\rrbracket r) \tag{2.31}$$

One important feature of these tensors is that they have the following property:

$$[\bar{\partial}, \eta^{-(p+1)}L_{a_0b_0,\dots,a_pb_p}^{(p)}] = 2(p+2)\eta^{-(p+2)}L_{ab,a_0b_0,\dots,a_pb_p}^{(p+1)}(\lambda\gamma^{ab}\lambda) \tag{2.32}$$

Note that the b operator, as well as the operators in the action, described in the following subsection, contains negative powers of η only. This generic property is not a priori obvious, and one can indeed find an alternative expression for the r -independent part of b :

$$b'_0 = \frac{1}{2}\xi^{-1}(\bar{\lambda}\gamma^a D)\partial_a \tag{2.33}$$

¹²This is a statement that deserves some moderation. As noted by several authors [44, 55], there is a “gauge invariant derivative”, $\tilde{\omega}_\alpha$, with $(\lambda\tilde{\omega}) = (\lambda\omega)$ and $(\lambda\gamma^{ab}\tilde{\omega}) = (\lambda\gamma^{ab}\omega)$. Replacing ω by $\tilde{\omega}$ in b_3 clearly leaves it unchanged, but allows for it to be shifted away as a $\bar{\partial}$ -exact term, since it is then gauge invariant without the leftmost factor $(\lambda\gamma^{ab}\lambda)$. This procedure can be continued until one indeed is left with a b operator constructed entirely out of the minimal variables, a procedure which however only has been performed explicitly for the b operator in $D = 10$. The singular behaviour is then replaced by singularities of the type $(N+a)^{-1}$ for some positive integers a . Such a minimal b operator turns out to contain a term with three fermionic derivatives, which is necessary for the correct fermion 4-point function [56]. Although the construction is interesting, we have not chosen to use it for further calculations.

This expression turns out to be q -equivalent to b_0 , but it does not, for some reason, allow for the analogous construction of a full b operator [57]. In ref. [47], where the non-minimal variables were first introduced, the topological aspects of the dependence on the pure spinor variables were stressed. There, one can interpret the topological property as the possibility to localise wave functions near the origin of the space of $D = 10$ pure spinors. Here, in contrast, the localisation seems to take place at the singular locus $\eta = 0$, which is the space of twelve-dimensional pure spinors. The deliberations of ref. [58] seem to aim at utilising this phenomenon.

Moreover, in constrast to the defining relation (2.29), we have not performed the full calculation in order to show that $b^2 = 0$. After having checked the vanishing of a number of terms, we have instead chosen to rely on the observation that there is no scalar cohomology represented by an operator σ such that a relation $b^2 = \{Q, \sigma\}$ could hold for $\sigma \neq 0$.

As is also discernible from eq. (2.29), the propagator cannot act on fields that are on-shell. Rather, it acts on off-shell fields produced from vertex operators and external fields. This brings us to the formation of the vertices of the theory, which of course is crucial for the amplitude diagrams.

2.3.3 The vertex operators R^a and T

In $D = 11$ supergravity, there is one more pure spinor field than ψ : ϕ^a , which starts with the diffeomorphism ghost and has the physical meaning of superspace geometry. Its cohomology coincides with the on-shell fields obtained by solving the linearised torsion Bianchi identities (after imposing conventional superspace constraints) [7, 8, 24, 25]. However, the two fields represent the same physical degrees of freedom, so that it is possible to express ϕ^a as $\phi^a = R^a\psi$: [33]

$$\begin{aligned} R^a &= \eta^{-1}(\bar{\lambda}\gamma^{ab}\bar{\lambda})\partial_b - \eta^{-2}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\bar{\lambda}\gamma^{cd}r)(\lambda\gamma_{bcd}D) \\ &\quad - 16\eta^{-3}(\bar{\lambda}\gamma^{a[b}\bar{\lambda})(\bar{\lambda}\gamma^{cd}r)(\bar{\lambda}\gamma^{e]f}r)(\lambda\gamma_{fb}\lambda)(\lambda\gamma_{cde}\omega) \\ &= \eta^{-1}(\bar{\lambda}\gamma^{ab}\bar{\lambda})\partial_b - \eta^{-2}L_{(1)}^{ab,cd}(\lambda\gamma_{bcd}D) - 6\eta^{-3}L_{(2)}^{ab,cd,ef}(\lambda\gamma_{ef}\lambda)(\lambda\gamma_{bcd}\omega) \end{aligned} \quad (2.34)$$

As for the b operator, we here have chosen a presentation which is not manifestly gauge invariant with respect to the pure spinor constraint. For an expression in terms of gauge invariant operators, see ref. [34].

The operator R^a obeys $[Q, R^a] = 0$ modulo terms of the form $(\lambda\gamma^a\mathcal{O})$, which allows for it to represent cohomology due to the fact that ϕ^a has an extra gauge invariance such that $\phi^a \approx \phi^a + (\lambda\gamma^a\rho)$ for an arbitrary $\rho(Z)$ [33]. Such “shift symmetries” are on equal footing with the pure spinor constraint and generic for non-scalar spinor superfields, such as ϕ^a . They were discussed in detail in ref. [36], and also occur for the fields of refs. [31, 32, 43, 59].

The physical meaning of ϕ^a , superspace geometry, is the clue to its role in the action, as a part of the vertices. There are several properties which make

$$S_3 \sim \int [dZ] (\lambda \gamma_{ab} \lambda) \psi \phi^a \phi^b \quad (2.35)$$

a good candidate for a 3-point coupling, which ought to be present in the full theory. To begin with, the quantum numbers match as R^a has dimension -2 and ghost number 2 . Moreover, the term $(\lambda \gamma_{ab} \lambda)$ is exactly what is necessary for the correct properties of antisymmetry and shift symmetry to be present [33, 55]. Also note that the 3-form potential of $D = 11$ supergravity is contained in the cohomology in ψ , while ϕ^a only contains it through its field strength 4-form. It was checked explicitly in ref. [34] that the Chern–Simons term $\int C \wedge H \wedge H$ is correctly produced from the present 3-point coupling, but even without this test, already the uniqueness of the deformation of the Pauli–Fierz action for linearised gravity [60] assures that a consistent non-trivial 3-point coupling will be that of $D = 11$ supergravity.

The requirement for the addition of this 3-point coupling, the full expression of which can be seen in eq. (2.2), to the free action is that the final action obeys the master equation. For this to happen, no more than one additional term is needed: the 4-point coupling with the operator T . More specifically, the condition $(S, S) = 0$ as in eq. (2.3) gives at hand a 4-point coupling in the action as in eq. (2.2) with the operator T : [34]

$$T = 8\eta^{-3} (\bar{\lambda} \gamma^{ab} \bar{\lambda}) (\bar{\lambda} r) (r r) N_{ab} \quad (2.36)$$

To be precise, it appears in the calculation of (S, S) through a commutator of two R^a operators as $(\lambda \gamma_{ab} \lambda) [R^a, R^b] = 3\{Q, T\}$. The most important feature of T is that it is nilpotent. We have that the factor¹³ $(\bar{\lambda} r) (\bar{\lambda} r)$ equals to zero, so T and thus any 4-point vertex cannot show up in an amplitude diagram more than once.

It is worth emphasising that the polynomial action thus obtained is the full action for $D = 11$ supergravity, and not some approximation. Typically, pure spinor superfield interactions tend to be of lower order than component ones. This has been observed in SYM [23], and in fact in all maximally supersymmetric models with action formulations [31–34]. A similar simplification from a non-polynomial component action to a polynomial pure spinor superfield action was shown to occur in the case of maximally supersymmetric Born–Infeld theory [36]. Non-polynomiality of a component action will arise on elimination of auxiliary fields. The action still enjoys the full gauge invariance encoded in the

¹³Note that the factor $(\bar{\lambda} r)$ shows up quite frequently in other expressions as well, for example in the propagator. Many terms (but not all) will therefore combine to zero, especially with a 4-point vertex present.

master equation, part of which constitutes diffeomorphism symmetry, local supersymmetry and tensor gauge symmetry. It is satisfying to see how many amplitude properties, otherwise obtained through more indirect methods, can be given a direct field theory interpretation.

Thus having concluded our description of the action of $D = 11$ supergravity presented in eq. (2.2), we now turn to how to put the formalism to use in relation to amplitude diagrams.

3 Amplitude diagrams

The remaining part of this paper concerns the construction and examination of amplitude diagrams in supergravity, with a starting point from field theory and expressed in the pure spinor formalism. Effectively, what might be used for this construction is determined by the presented theory. External fields must be represented by on-shell pure spinor superfields ψ with components of ghost number 0, 3- and 4-point vertices ought to be described in full by the action in eq. (2.2) and each propagator ought to contain b/p^2 . What other features might be needed must be covered by the operators and variables either present in the theory or possible to introduce into it, δ -functions and integrations $[dZ]$. Perhaps the most subtle and powerful feature of the formalism is that so called regulators, Q -exact terms, may be added through the insertion of $e^{\{Q, \chi\}}$, as described in subsection 2.2.

We will begin this section by performing a thorough identification of how to describe tree parts of amplitude diagrams in terms of the ingredients described above. Part of this incorporates the concept of regularisation with respect to the divergences such integrands may present, a procedure which is required in order for finite results to always be obtainable. As in the previous section, it should be noted that this last procedure is well known, though the projector used for the general regularisation (in this formalism) has not been presented before, to our knowledge.

Following that, we will introduce loops and the regularisation which every loop must be subject to in order for general amplitude diagrams to have the slightest possibility to be finite — a regularisation already known from SYM in a first quantised version [12, 40], which in refs. [11, 12] was generalised to supergravity as well. The broader question of finiteness is to be (partly) addressed in the next section, but the general properties will be sorted out in this one.

The last subsection concerns features of the amplitude diagrams, beginning with the ways loop momenta split between different loops and why bubbles and triangles cannot show up in the amplitude diagrams, in consistency with previous results. We end by some notes on how to deal with components exterior to a loop

structure¹⁴ in relation to the very same loop structure, before we move on to the next section, which concerns an actual examination of the loop amplitudes, with respect to their UV behaviour.

3.1 Building blocks for tree diagrams

The tree components are mostly given by the form of the action in eq. (2.2). Beginning with the vertices, they basically are to be read straight off that same equation. By construction, they cannot be anything else, and they fulfil all required properties. The propagator on the other hand needs a slight change in order to fit in, and then there really are no other parts of the tree diagrams but the general regularisations, which will be described in the next subsection. Our main concern is the ultimate ultraviolet behaviour of the last remaining momentum integrals, so we will not care about details concerning overall normalisation of amplitudes, neither combinatorial factors of diagrams nor absolute normalisations of integrals over pure spinor space.

To begin with, the 3- and 4- point vertices connect 3 and 4 fields respectively, where each field is of ghost number 3. At the same time, amplitude diagrams must be physical in the sense of them having ghost number zero, as the action. The latter (and dimensional properties) is what shapes the action and the vertices so that the parts which are not fields ψ have ghost number -9 and -12 respectively. Part of both types of vertices are two pure spinors and two R^a -operators, each of ghost number -2 , acting on the connected fields. There must also be an integration $[dZ]$, of ghost number -7 , quite in analogy with ordinary field theory where each vertex contains an integration over all of space. This describes the 3-point vertex in full up to a number times the coupling constant, which is given by the action and kept implicit here. The 4-point vertex in addition contains T , acting on one of connected fields just as the R^a 's. For an illustration of how this works, see figure 1. What is important to remember is that not two of the operators can act on the same field and give a non-zero expression, but once that requirement is fulfilled, they can be taken to act on any of the connected fields. The end result is the same, a property which originates from partial integration in the action (the operators R^a and T are first order in derivatives) and the fermionic statistics of the field ψ .

Moreover, a field needs to be outside of the cohomology in order to propagate, so a propagator must be attached to two vertices. This means that in order for all types of amplitude diagrams to end up with ghost number zero in total, as

¹⁴By a loop structure we here denote a loop or loops, sharing propagators or connected by such. Any part which is attached to *one* loop but does not contain or constitute a part of any loop itself, is not considered as a part of the loop structure. Rather, we term it a tree part connected to the loop structure by an “outer leg”. This invariably ends in one or several external legs.

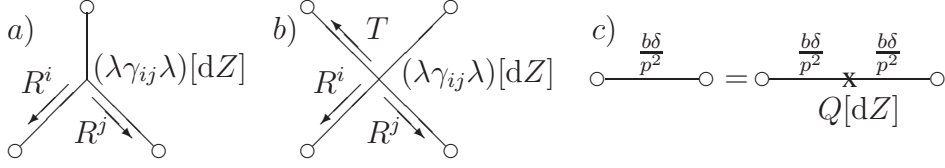


Figure 1: The three illustrations above show the building blocks of the tree diagrams. The 3-point (a) and 4-point (b) vertices are depicted with their components, some of which in supergravity act on the fields which are to be attached. They can be on-shell, representing external legs, which is not true for the propagator (c). Note also that two propagators can be connected with Q since $\{Q, b\} = p^2$ and $\{b, b\} = 0$, which gives at hand one propagator as depicted in (c). It is implicit in the notation on the RHS of (c) that the two propagators and the Q are formed out of three different sets of variables, brought together by $[dZ]$'s and δ -functions.

should be, the total propagator which connects different parts of amplitude diagrams must have ghost number 6. That way, each end of it can be considered to have ghost number 3, as a field ψ connected to a vertex would. This fits with the total propagator consisting not only of b (of ghost number -1) and the factor $1/p^2$, but also a δ -function, as depicted in figure 1, since the δ -function (with respect to the measure $[dZ]$) must have ghost number 7, as the ghost number of $[dZ]$ is -7 . However, the presence of the δ -function is in itself perfectly physical, as the b/p^2 propagates the field coming from a vertex through superspace, to another vertex. The connection is made by the δ -function, which makes the propagator ultralocal in the pure spinor variables.

The components presented above suffice for the construction of tree diagrams, as illustrated in figure 2. At the introduction of loops, a special regularisation is needed, which will be treated in subsection 3.3. Note though, that in order to be finite, amplitude diagrams generally also need to be regularised with respect to the pure spinors, as will be presented in the next subsection.

3.2 General regularisations

At an integration over pure spinor space, a general integrand typically threatens to diverge in two different ways due to the presence of the pure spinors λ and $\bar{\lambda}$, as described in subsection 2.2.2. The first arises from $\xi \rightarrow \infty$. The second contains the two different ways in which η may approach zero (at the origin or at the singular submanifold of twelve-dimensional pure spinors), so that negative powers of ξ or η diverge. The operators previously introduced typically contain negative powers of η , which means that the first divergence to occur is at the singular submanifold. This is the main cause of divergent integrals.

Both types of divergences and regularisations are known from maximal SYM

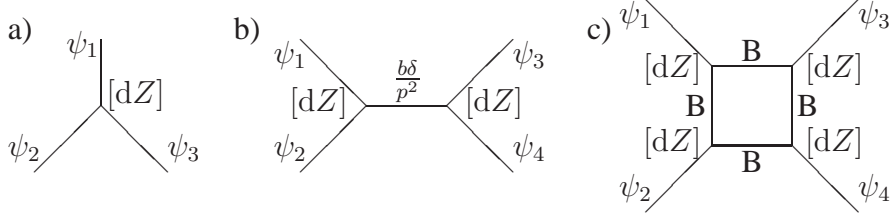


Figure 2: Some of the most basic tree and loop diagrams are the 3-point vertex (a) and the 4-point interaction amplitudes with none (b) and one loop (c) respectively, where the vertex operators have been left implicit. This shows the basic principles of connecting the constituents depicted in figure 1 to each other and to external fields. In diagram (c), B denotes the total propagator $b\delta/p^2$. More usually though, the factors of $1/p^2$ are kept implicit, as well as integration measures and δ -functions, which after all are integrated out before long.

[12, 40], and the procedure differs very little, despite the differences in the singular submanifolds. The first type of divergence is regularised with the introduction of a regulator:

$$e^{\{Q, \chi\}} = \left[\chi = -\bar{\lambda}\theta \right] = e^{-\lambda\bar{\lambda} - r\theta} \quad (3.1)$$

This not only ensures a good convergence in the limit of infinity for the bosonic spinors, but also saturates the fermionic integrals of $[d\theta][dr]$. The regularisation, or a similar one, should be used already in the classical action in order to produce sensible results for component fields. The bosonic part of the regulator has a simple and attractive interpretation, where fields are considered as sections of line bundles over the Kähler manifold of pure spinors. This line bundle, a so called prequantum line bundle (see e.g. ref. [61]), is equipped with a metric related to the Kähler potential, which here is chosen to be $(\lambda\bar{\lambda})$, the metric obtained from embedding the pure spinor space in a flat $32_{\mathbb{C}}$ -dimensional spinor space¹⁵.

The latter divergence is superficial as well, provided that the operators involved are well behaved enough. To begin with, note the conditions on an integrand in order for it to be finite, listed in eq. (2.26). If an operator obeys those conditions, it can be shown that it is BRST equivalent with an operator which is not singular with respect to (ξ, η) , as in ref. [40] for SYM. Moreover, since we only are interested in BRST-equivalent classes of operators (the key feature of the introduction of regulators in subsection 2.3.1), the singular operator can be exchanged for its regular BRST-equivalent.

¹⁵This is not the Kähler potential corresponding to the Calabi–Yau metric [48], but that is unimportant since the deformation is Q -exact. It is tempting to imagine that this connection to geometric quantisation could be continued, and that a non-commutativity on pure spinor space could provide natural regularisation also of other potential divergences. We have not managed to show that this happens.

The principle is that an integration over $[dZ]$ (without $[dx]$) can remove the negative effective powers of ξ and σ in the operator, so a new set of variables (with the same properties as the first) is introduced and integrated over. The result is an operator dependent on the non-minimal variables, which does not contribute to any divergence at the origin or at the singular submanifold. As the procedure can be performed for any number of operators, and all operators fulfil eq. (2.26), each and every integrand has a representative of its orbit under BRST equivalence which satisfies eq. (2.26). That is why this procedure, described below, regularises the divergence with respect to negative powers of (ξ, η) .

A sketch of the procedure is as follows. With the introduction of two constant pure spinors f_α and \bar{f}_α (corresponding to a new set of λ_α and $\bar{\lambda}_\alpha$) a change of the operator might be performed: [40]

$$\begin{aligned} O(\lambda, \bar{\lambda}) &\rightsquigarrow \int [df][d\bar{f}] e^{-f\bar{f}} e^{i\varepsilon(fW + \bar{f}\bar{W})} O(\lambda, \bar{\lambda}) = \\ &= \left[\lambda' = e^{i\varepsilon fW} \lambda, \bar{\lambda}' = e^{i\varepsilon \bar{f}\bar{W}} \bar{\lambda} \right] = \\ &= \int [df][d\bar{f}] e^{-f\bar{f}} O(\lambda', \bar{\lambda}') \end{aligned} \quad (3.2)$$

Here, $(\lambda', \bar{\lambda}')$ are pure spinors just as $(\lambda, \bar{\lambda})$ respectively. Since the operator diverges no more than is taken care of by $[df][d\bar{f}]$, it would not contribute any negative factors of ξ or σ to an integrand. [40]

For such a construction to be possible, suitable “translation” operators W_α and \bar{W}_α must be available, and the process must be possible to describe through the introduction of a new regulator. To begin with the operators, the W ’s represent gauge invariant versions of the ω ’s. If the ω ’s were to be used instead of the W ’s, eq. (3.2) would incorporate $\lambda' = \lambda + \varepsilon f$, which is not necessarily a pure spinor. For that to be true, a certain gauge must be chosen, with specific constraints on combinations of for example ω and \bar{f} , but the equations need to be independent of the choice of gauge. That is what calls for the construction of derivatives that are obtained using projectors on the cotangent space of pure spinors in $D = 11$:

$$W_\alpha = P^\beta{}_\alpha \omega_\beta : \quad P = \mathbb{I} - P_\perp \quad (3.3)$$

The reason for this expression for the projection $P^\alpha{}_\beta$ is that it acts as the identity on tangent vectors $d\lambda^\beta$ which fulfil $(\lambda\gamma^a d\lambda) = 0$. Essentially, the pure spinor constraint is enforced by the removal of the parts which would not automatically obey it, by the presence of P_\perp . The construction of W is far more complicated in

$D = 11$ than in $D = 10$. It is presented in detail in appendix C:

$$\begin{aligned} P^\alpha{}_\beta &= \delta^\alpha{}_\beta - \frac{1}{4\alpha}(\gamma_a \bar{f})^\alpha (\lambda \gamma^a)_\beta - \frac{1}{2\alpha\beta}(\gamma_a \bar{f})^\alpha (\lambda \gamma^{ai} \lambda)(\bar{f} \gamma_{bi} \bar{f})(\lambda \gamma^b)_\beta + \\ &+ \frac{3}{4\alpha\beta}(\gamma_{[ij} \bar{f})^\alpha (\bar{f} \gamma_{kl} \bar{f})(\lambda \gamma^{ij} \lambda)(\lambda \gamma^{kl})_\beta \end{aligned} \quad (3.4)$$

Here, $\alpha = (\lambda \bar{f})$ and $\beta = (\lambda \gamma_{ab} \lambda)(\bar{f} \gamma^{ab} \bar{f})$. Moreover, with a short notation according to $P^\alpha{}_\beta = (\gamma_a \bar{f})^\alpha R^a{}_\beta(\lambda, \bar{f})(\lambda \gamma^b)_\beta$, the above gives at hand a “translated” pure spinor:

$$\lambda' = e^{(\varepsilon W)} \cdot \lambda^\alpha = \lambda^\alpha + \varepsilon^\alpha - \frac{1}{2}(\gamma_a \bar{f})^\alpha R^a{}_\beta(\lambda + \varepsilon, \bar{f})((\lambda + \varepsilon) \gamma^b (\lambda + \varepsilon)) \quad (3.5)$$

Analogously, the form for \bar{W} is found by a change of all barred and unbarred variables: $\lambda \leftrightarrow \bar{\lambda}$ and correspondingly for all other indices which exist in both versions [40]. What is important for the continuation of the description of the regularisation procedure is that the correct, gauge invariant derivatives can be constructed.

In order to describe the process through the introduction of a new regulator, a full set of variables must be introduced. Not only the pure spinors are needed, but also constant fermions g_α and \bar{g}_α with constraints: [40]

$$g \gamma^a f = \bar{g} \gamma^a \bar{f} = 0, \quad [Q, f] = \{Q, \bar{g}\} = 0, \quad [Q, \bar{f}] = \bar{g}, \quad \{Q, g\} = f \quad (3.6)$$

This can be compared to the introduction of the non-minimal variables, though the unbarred variables imitate (λ, θ) instead. With¹⁶ $V = \bar{W}(\bar{\omega} \rightarrow s)$ we then have that $\{Q, gW + \bar{f}V\}$ contains a term $fW + \bar{f}\bar{W}$, as well as that $\{Q, \bar{f}g\}$ contains a term $f\bar{f} + g\bar{g}$, which means that it is possible to do a construction as in eq. (3.2). In fact, the correct regularisation for an operator (acting on fields etc.) is: [40]

$$O(z) \rightarrow \int [df][d\bar{f}][dg][d\bar{g}] e^{-\{Q, \bar{f}g\}} e^{i\varepsilon\{Q, gW + \bar{f}V\}} O(z) e^{-i\varepsilon\{Q, gW + \bar{f}V\}} \quad (3.7)$$

This takes care of the divergences of the operator in question, if the singularity is not too bad (see below). It also performs another regularisation, necessary due to the introduction of the integration over the new variables: the first type of regularisation discussed above, for the set of variables represented by (f, \bar{f}, g, \bar{g}) .

In total, we have a procedure that provides a smoothing of a singular operator. Performed once, it takes care of singularities within the limits of eq. (2.26). If that is not sufficient, the procedure can be performed over and over and again. Each time, new variables are introduced and the smoothing of the operator or integrand

¹⁶Here $\bar{\omega} \rightarrow s$ denotes that every $\bar{\omega}$ in the expression is replaced by an s .

corresponds to a smoothing of the maximally allowed singularity in eq. (2.26). Eventually, this gives at hand a regular expression. [40]

The number of times this procedure must be performed is roughly determined by how high a negative power of η there is present in the expression in question, as it is the only scalar that shows up with negative powers in the operators. As such, the power of σ represents the worst singularity, with “one smoothing procedure” taking care of no more than $\eta^{-(11/2)}$. A general regularisation is therefore typically very intricate. The geometrical interpretation though, can be compared to a “heat kernel” regularisation [40]. It can most certainly be replaced by a more geometric construction explicitly involving smoothing by the Laplacian on the pure spinor space. This would involve further geometric investigations along the lines of ref. [48], but it is not clear that it would provide any calculational advantage, especially since it would not respect the splitting into separate holomorphic and antiholomorphic factors of the present approach.

3.2.1 The interpretation of regularised amplitudes

The above establishes that the proposed regularisations work. But in what way do they show up in a typical integrand, originating in e.g. an amplitude diagram?

Worth to note about regularisations in general is that they can be performed in random order and at any time up until the properties which they alter are being used. For the general regularisations described above, this is not much of a restriction, since they do not meddle with an expression very much, in contrast to the loop regularisation which will be introduced below. The regularisation in eq. (3.1) contains no derivatives, and therefore changes none of the variables that were present before the introduction of the regulator, while the variables the regularisation method described in eq. (3.7) acts on are replaced by variables of very similar (indeed, partly identical) properties. Furthermore, the introduced variables exist in a very limited region (one operator) and are integrated out, properties which combine in such a way as to restrict the impact of the alteration to one thing only: the regularisation of the expression in question.

In specific, this means that the general regularisations can be inserted at any time up until the integration over the superspace variables, yielding the same result. As such, the regularisations due to the pure spinors can be kept implicit in the examinations of the amplitude diagrams, and as a rule they are never mentioned.

However, not all regulators are as well behaved as the above, which will be noted in the next subsection. The regularisation which is necessary to introduce due to the presence of loops acts on the variables in a slightly different way, exchanging one for another. In contrast to the regularisations above, that means that the regularisations must be performed at a certain time — before properties of different variables are being used.

3.3 Loop regularisation

At the introduction of loops, in contrast to a description with tree diagrams only, a problem with the phase space field theory construction typically occurs: it lacks the description of the freedom of momenta etc. that is introduced by the formation of a loop. This occurs when the propagator is too local with respect to one of its bosonic variables, e.g. a function $\delta(x - y)$ in space. At the integration over the vertices attached to a loop, compare figure 2, the last integration then must be performed over $\delta(0)$, which diverges. This last expression, present when one integration remains, is what is interesting in the examination of amplitude diagrams, and something divergent does not make any sense.

Ordinarily, this problem does not show up due to the presence of a kinetic term in the theory. As such the propagator in the momentum space is expressible in terms of an exponential of the Laplacian, instead of a factor 1, and the phase space propagator is not too local, but a Gaussian curve:

$$\frac{1}{p^2} \sim \int_0^\infty e^{-ap^2} da \quad \leftrightarrow \quad e^{-a(x-y)^2}, \quad a > 0 \quad (3.8)$$

The above is also the case for the variable x in maximal supergravity described with the pure spinor formalism, but only because of the gauge fixing, which gives at hand the relevant dynamics (b/p^2). The theory contains two more bosonic variables $(\lambda, \bar{\lambda})$ without corresponding kinetic terms. Moreover, the propagator is too local with respect to them, represented by δ -functions, and needs to be changed into a less local one in order for the theory to make sense. One possible way to do that is to deform the theory by introducing suitable kinetic terms, giving at hand properties for the propagator as in eq. (3.8) with respect to the pure spinors as well as x . However, such a deformation is not desirable, as the theory would be deformed as well, and there would be no way of connecting results provided by it to results in maximal supergravity.

Instead, the key feature to be recognised is that the pure spinor formalism gives at hand a possibility to *imitate* the construction in eq. (3.8), through the use of BRST-equivalent terms. Either a suitable regulator is added, a process which bears a striking similarity to the expression on the left hand side in eq. (3.8), or else the b operator might be changed into something less local, yet still with the property $\{Q, b\} = \partial^2$. Such a conversion, in either direction, can of course be done for any number of propagators in a loop diagram. In essence, it means that it is possible to exchange the too local propagator in phase space for a less local one. The result is a remaining integration over a function which is finite, and the problem is solved.

A possible observation at this point is that the general regularisation described in eq. (3.7) performed on a b operator might do for the description of a not too local

propagator. However, that regularisation is quite intricate and it is far from obvious how to extract any loop properties with such a construction. At the end of the day, another solution is sought for, with which properties of amplitude diagrams might be observed before the final regularisation in eq. (3.7).

A Q -invariant alteration of the b operator into something less local, still obeying $\{Q, b\} = \partial^2$, might be a convenient way of finding a solution. The construction of eq. (3.7) with $O \rightarrow b$ seems to provide such a deformation, but is extremely impractical to use in loop calculations. We have put some effort into trying to find alternative explicit expressions for b , not involving extra variables, but have not succeeded. Although we still do not want to exclude the possibility of finding a more efficient, non-local but regular, propagator; given the present status of the propagator, what remains is the introduction of one more regulator.

3.3.1 Loop regularisation in superspace

In superspace, the only apparent way of regularising a loop, while retaining any possibility of extracting results, is to recognise the freedoms of momenta etc. in the loop, instead of using the last δ -part of the propagator, in each loop, which would diverge. In figure 1 the latter corresponds to gluing two connected propagators together, so that there only is one δ -function which by the fusion turns into “ $\delta(0)$ ”. But instead of using this last δ -function, which is superfluous and does not capture the loop properties, all the momenta in the loop (of number I) are changed so that the freedoms of the loop momenta are described: $\partial \rightarrow \partial + \partial^I$ for the terms in the loop in consideration, and the corresponding for D^I , N^I , N_{ab}^I , \bar{N}^I , \bar{N}_{ab}^I , S^I and S_{ab}^I . Each of these “coordinates” of the cotangent space represents a new variable, ranging in value within its domain, so it is also necessary to introduce the corresponding integration measures, which implies that the process corresponds to the following: [12, 40]

$$\int f(b, p) \delta(0) [dZ] \rightarrow \int f(b_{new}, p + p^I) [dZ] d^D p^I [dD^I] [dN^I] [d\bar{N}^I] [dS^I] \quad (3.9)$$

Here, b_{new} is such that each b depends on the momenta and the loop momenta of the loop that the propagator is part of¹⁷ [12]. The last three integration measures are schematically given as¹⁸:

$$[dN^I] \sim \lambda^{-16} d^{23} N^I, \quad [d\bar{N}^I] \sim \bar{\lambda}^{-16} d^{23} \bar{N}^I, \quad [dS^I] \sim \bar{\lambda}^{16} d^{23} S^I \quad (3.10)$$

¹⁷All loops need to be regularised, so when a propagator is shared between different loops, the b -ghost depends on the original momenta and the momenta of the loops: $\partial \rightarrow \partial + \sum_I \partial^I$.

¹⁸ N here temporarily denotes both N and N_{ab} , and the initial integration is with respect to ω , which however must show up in the gauge invariant quantities described by the N ’s. The corresponding is true for \bar{N} and S .

The powers of λ and $\bar{\lambda}$, which are only given schematically, follow from the simple observation that while $[d\lambda] \sim \lambda^{-7} d^{23} \lambda$, the inverse metric governs the integration over cotangent space, which then becomes $[dN] \sim \lambda^7 d^{23} \omega \sim \lambda^{-16} d^{23} N$.

However, the amplitudes are independent of S^I , S_{ab}^I , \bar{N}^I and \bar{N}_{ab}^I , the forms of which are described in eq. (2.6), as these quantities do not show up in any expression. They certainly are not present in any term which is not a regulator, nor in the regularisation presented in eq. (3.7) due to the form of it. Moreover, the absence of the two last quantities gives at hand that the amplitudes cannot depend on N_{ab}^I or N^I either [12], so a regulator is needed in order for the integration to make sense, $e^{\{Q, \chi\}}$ [12, 40]:

$$\begin{aligned} \{Q, \chi\} &= [\chi = -k(NS + N_{ab}S^{ab}), \quad k > 0] = \\ &= k \left((\lambda D)S + (\lambda \sigma_{ab} D)S^{ab} - N\bar{N} - N_{ab}\bar{N}^{ab} \right) \end{aligned} \quad (3.11)$$

Note that the components of D^I in each D etc. here have been kept implicit, as is most convenient. As long as the free loop momenta have not been integrated out, they are present in every momentum term.

The expression above brings us back to the interpretation of the regularisation as originating in the introduction of a regulator. It has certain properties which will be discussed in more detail below, such that an r can be turned into “ $\lambda D \bar{\lambda}$ ” for example [12], but for now, we continue the examination of the impact of the regulator on the newly introduced integrations. As it provides the only terms that the integrations in eq. (3.10) concern, those integrations (with the regulator and all other terms implicit) give at hand [12]:

$$\int [dN^I][d\bar{N}^I][dS^I] \sim \frac{1}{\lambda^{16}\bar{\lambda}^{16}} \int [dS^I] \sim \lambda^7 (D + \sum_I D^I)^{23} \quad (3.12)$$

From this we can see, for example, that in order for the free, fermionic momenta of S to be integrated out, the terms need to be taken from the regulator, which brings down an extra term “ λD ” for each S that needs to be integrated out. The end result is furthermore quite convenient as a factor λ^7 has been provided for the volume form $[d\lambda] \sim \lambda^{-7} d^{23} \lambda$ in $[dZ]$. However, as there are 32 spinors D^I for each loop, the result for the entirety of the newly introduced integrations is:

$$[dN^I][d\bar{N}^I][dS^I][dD^I] d^D p^I \stackrel{(3.12)}{\sim} \lambda^7 d^9 D^I d^D p^I \quad (3.13)$$

The effective integration thus requires 9 D^I ’s per loop to be extracted from the amplitude diagrams in order for a non-zero result. These must be provided either through the D^I ’s already present in the propagators and vertices, or through the use of the regulator.

The last part of eq. (3.13) is the integration over the momenta p^I present in the loops, which is done up to some momentum cut-off limit, Λ . With respect to the finiteness of the theory, these last integrations need further examination. Part of this, more specifically the UV behaviour of the amplitude diagrams, will be discussed in the next section.

All in all, the introduction of the momenta freedoms and their integration measures changes the integration over each loop so that a finite result can be obtained. The procedure corresponds to an integration over the cotangent space to a point in superspace, described with the real metric, which is Calabi–Yau.

3.3.2 The matter of loop regularisation

As noted for the general regularisations, regulators can be introduced randomly up until their properties, or the properties they affect, are used. The latter is what makes the loop regularisation a bit trickier than the general regularisations.

The apparent condition brought on by the regularisation is that 9 D^I ’s per loop need to be extracted from the integrand. This in itself is rather harmless, though crucial for the amplitude properties. The complication is brought on by that r can be converted into D :

$$r_\alpha \rightarrow \begin{cases} (\lambda D)\bar{\lambda}_\alpha \\ (\lambda\gamma_{ab}D)(\bar{\lambda}\gamma^{ab})_\alpha \end{cases} \quad (3.14)$$

The above calls for two different observations. Firstly, the first transformation cannot be applied to any r which is a part of a structure as the one described in eq. (2.31). Due to symmetry properties, the result would be zero. That is why the first transformation only can be applied to r ’s part of the operator T in the 4-point vertex, and only after the second transformation has been used at least once, at that.

Secondly, the above clearly changes the algebraic properties of the indices attached to the r ’s which are to be transformed. Unless the indices, naively described as $r_\alpha r_\beta$ “before” regularisation, are part of structures that ensure them to be antisymmetrised regardless of the presence of the r ’s, such as individual operators, they are not necessarily restricted to be antisymmetrised. Then, one has to take the regularisation into consideration before using the properties of the non-minimal variable r . In short, the constituent part need to be regularised before they are put together.

The simplest solution to the above might have been not to use the property of the regulator which changes variables according to eq. (3.14). Indeed, where that is possible, that is what we will do. It is, as described above, quite all right to choose to examine any term in the expansion of a regulator (as long as the constant one is kept, of course); the answer will be correct, provided that it is not nonsense like $0 \times \infty$. However, without the use of the regulator, all terms with a factor r^x

with $x > 23$ would incorrectly give at hand a zero contribution, due to the fact that one $\bar{\lambda}$ invariably is part of the integrand. The condition $(\bar{\lambda}\gamma^a r) = 0$ then reduces the degrees of freedom of the fermionic spinor r to 23. But the fact that the r cannot show up in any higher powers than 23 does not prevent the presence of “converted” r ’s. So all terms with r to a higher power than 23 need to be brought into a form with r^{23} in order to be interpreted correctly, a procedure which is carried through with the help of the loop regulator.

To be precise, one may note that r and D in a certain manner can be treated on an equivalent basis. Not only due to the transformation property of eq. (3.14), but due to the fact that the presence of an r in an operator only comes at the expense of a D , compared to the terms with lower powers of r , compare e.g. eq. (2.30). At the examination of whether or not an expression needs to be regularised with respect to r , one might therefore consider only the terms solely consisting of r ’s, without the presence of any D , and note that a regularisation with respect to r only gives a non-zero result if the power of r present exceeds or equals $23 + 9L$. This allows for the r^{23} remaining after the conversion and the $(D^I)^9$ required for each loop integration in order for a non-zero result, and provides a most convenient way of determining when there might be too many r ’s present in an expression.

Moreover, note that the r ’s remaining after the loop regularisation are distributed across the former positions of r . Where they sit does not alter any properties of the expression. This is on equal footing with the fact that if part of a loop structure, which on its own does not need to be regularised, displays certain properties when it is not regularised, those same properties will be present when it is regularised as a part of the entire loop structure. We have BRST invariance, so the properties obtained are the correct ones. The converted r ’s need only be taken into account when properties dependent on structures with too many r ’s present are being examined.

3.4 Amplitude characteristics

The loop regularisation and the formation of loops are the key features restricting the constituent parts of the loops and in what ways they can be put together into amplitude diagrams with tree parts. The first condition, 9 D ’s to be extracted from each loop, is fairly straightforward. The conversion of r ’s into D ’s and the fact that loop operators are allowed to act in all directions on the other hand unsurprisingly gives at hand that even the description of a diagram with only a few loops is quite intricate. The most apparent properties will be discussed below though.

To begin with, we may take note of the different parts of an amplitude diagram. There are two different kinds: planar and non-planar diagrams, the former which can be drawn on a plane without ever crossing a line. That is a property which refers to the composition of the loop structure. In the general scheme of things

though, tree parts are connected to a loop structure via a number of so called outer legs, as are the external legs that do not connect to tree parts. Moreover, although some diagrams, tree diagrams, do not contain a loop structure, those will not be treated below, as no surprising restriction on the properties thereof occurs.

What we will begin by discussing is the loop structure itself, starting out with the most simple case, the one-loop amplitude. An examination of it shows why bubbles and triangles are absent in amplitude diagrams. This naturally carries on towards a discussion on the splitting of momenta between different loops, particularly with respect to the presence of outer legs, and to some degree the conditions on non-planar diagrams also. It is, however, difficult to discern properties of structures which require regularisation with respect to r . The question of when such regularisations are required is addressed in the next section, on the UV characteristics of the amplitude diagrams.

Subsequently, we discuss the picture of the loop structures which has emerged, and finally, the general conditions on the amplitudes are taken into account, with respect to the “outer components” that are not part of the loop structure. In total, this gives at hand a fairly detailed, general description of the amplitude diagrams.

3.4.1 No bubbles or triangles

A known feature of amplitude diagrams in maximal supergravities is that they cannot contain any bubbles or triangles (loops with less than four vertices attached) [12, 37, 38, 62]. In the formalism here presented, this is a consequence of the fact that each loop for a non-zero result needs to contribute 9 D ’s to the integration over the loop variables. As is shown below, this cannot be provided by a bubble or a triangle, no matter what their outer legs are connected to, so no bubbles or triangles can exist as parts of a loop structure.

The reasoning goes as follows. To begin with we address the question of whether or not loop regularisation with respect to r will be necessary, compare subsection 3.3.2, which it turns out not to be. Each set of vertex and propagator operators¹⁹ contains at most r^7 , or r^{10} (for the 4-point vertex), yielding that the highest possible power of r present in a one-loop amplitude with less than 4 vertices is r^{24} . Although this power is greater than 23, that term is accompanied by no D and cannot give at hand $r^{23}D^9$ from regularisation, and as such vanishes. Effectively, a bubble or a triangle contains no terms with r^x , $x > 23$ on its own. Thus, as $e^{\{Q,\chi\}}$ is Q -exact, the exponential term does not determine whether or not a bubble or a triangle exists, and it may be treated as a factor 1. As such, r ’s

¹⁹A set of vertex and propagator operators here denotes the presence of one vertex and one propagator, i.e., two R^a ’s and one b in the case of a 3-point vertex, and the additional T in the case of a 4-point vertex. A bubble consists of no more than two such sets and a triangle of three, although the 4-point vertex only can show up once.

cannot be converted into D ’s, and a propagator or a vertex can at most contribute with two D ’s to the loop integration.

This would be enough to exclude bubbles, but not triangles. However, each unregularised vertex cannot contribute with more than one D to one and the same loop integration, since the antisymmetrisation of the indices in R^a and R^b ensures that the indices of their two D ’s are symmetrised, whereas they need to be the opposite of that. Furthermore, the part of the propagator which might contribute with two D ’s looks like $f(\lambda, \bar{\lambda}, r)^{ijk} D\gamma_{ijk} D$, compare eq. (2.30). This quantity is fermionic, which gives at hand that in order for two propagators to both contribute with two D ’s to the loop integration, those two parts must be antisymmetrised. At the same time, the four D ’s must be antisymmetrised, as the 9 D ’s are. In order for the latter condition to be true, the two factors of $D\gamma_{ijk} D$ need to be symmetrised, in which case the former condition is such that the two factors of $f(\lambda, \bar{\lambda}, r)^{ijk}$ need to be antisymmetrised. This is an apparent contradiction as the sets ijk and $i'j'k'$ cannot be both symmetrised and antisymmetrised at the same time, with the consequence that two propagators cannot both contribute with two D ’s to one and the same loop integration.

In total, only one propagator can contribute with D^2 to one and the same loop integration, the rest of the propagators as well as the vertices cannot contribute with more than one D each. This renders bubbles non-existent, since they at most can contribute D^5 to the loop integration. The same goes for the triangle, which at most can contribute D^7 to the loop integration. For 9 D ’s, at least four legs need to be attached to the loop.

Any one-loop amplitude thus must have at least four external legs. Moreover, with the analogy observed in subsection 3.3.2, every loop in an amplitude diagram must have at least four vertices, so neither bubbles nor triangles exist in maximal supergravity.

3.4.2 The splitting of momenta between loops

It is important to note, that the absence of bubbles and triangles not always is enough to render an amplitude diagram non-zero, with respect to the 9 D ’s that need to be claimed for each loop. For example, if a propagator is shared between several loops (possibly more than two if the diagram is non-planar), its (absolute) maximum contribution of 3 D ’s to the surrounding loops might fall short of what is needed. In such a case, there would be a requirement for one or several of the outer legs to be attached to some of the loops in question in order for the diagram to be non-zero. As each outer leg adds a propagator to the loop to which it is attached, the D ’s that propagator could provide to the loop integration would ease the requirement on the contribution from the first propagator, otherwise shared between too many loops.

This brings us to a more detailed examination of how the momenta in a loop can take values in the degrees of freedom introduced by the loops, as well as the original momenta, which is another feature of the loop regularisation. This is represented by e.g. $\partial \rightarrow \partial + \sum_I \partial^I$ with I summed over the different loops the original momenta constitute a part of. To simply assume that these types of momenta end up distributed randomly, even taking into account that 9 D^I 's must be removed for each loop integration, is to pursue matters a bit too far though. Some terms vanish.

Consider, for example, the case of a one-loop amplitude with N outer legs, or any part of a loop which can be constructed from one propagator to which $N - 1$ outer vertices are attached. It has N propagators that all, naively, have momenta: $\partial + \partial^I$. They are all a part of the same loop, and no other, so even the possible term $\sum_J \partial^J$, for the general case, can just as well be regarded as a ∂^I . However, a b operator placed between two certain vertices, called number 1 and 2, can act in any direction since it is a part of a loop. The way it acts on or is acted on by the amplitude diagram and its loop regularisation is distributed across the entire structure. In fact, the description is equivalent to one where that b operator is not placed between vertex number 1 and 2, but instead placed on any of the other propagators, right next to the b operator which was there in the initial description. This is perfectly fine²⁰. Moreover, the two terms contain at the most r^6 , so the ensuing structure can be analysed before any regularisation with respect to r . For example, some combinations of those two b operators depend entirely on either the momenta ∂ , or the loop momenta ∂^I , in which case we have a factor $\{b, b\} = 0$ so that the term vanishes.

Let us be explicit. If an operator $D\gamma^{abc}D$ were to be part of *one* loop regularisation, it would effectively contain $D\gamma^{abc}D + 2D^I\gamma^{abc}D + D^I\gamma^{abc}D^I$. This happens to the b operators as well, and because of the property $\{b, b\} = 0$, two propagators which are part of the same loop structure cannot be dependent entirely on the same set of momenta ∂^I , with I taking the value of a loop, or not. Most propagators are split between the momenta of different loops, as the term $D^I\gamma^{abc}D$ is.

In specific, in the setting above, only one of the b operators can depend entirely on the loop momenta ∂^I . The rest at the very least contains one D each, a property which will come in handy in the next section, on the UV behaviour of amplitude diagrams. In relation to the previous subsection, this is also the fundamental property that explains why no more than one of the propagators in a bubble or triangle can contribute with two D 's to the loop integration.

²⁰Note that this is not equal to having two b operators on the same propagator in the picture where the loop momenta are absent. Such a term would yield $\{b, b\} = 0$. However, this is not true when the loop momenta are present, and thus b operators can be brought onto the same propagator through partial integration, with a non-zero result.

Moreover, the examination can be drawn a bit further. We note that the number of propagators that can be compared in this way, at the same time, without any transformation of r , is at most 7. Out of these, one may depend on ∂^I , another on ∂ , but the rest must be split between the two sets of momenta, and the entire expression for $\{b(\partial + \partial^I), b(\partial + \partial^I)\}$ can be investigated. It is non-zero, but only for terms that in combination have r^3 as a part of them. In specific, the remaining components originate in b_1 and b_2 and consists of terms with D^3 split between the momenta. When r is not regularised, the propagators form no terms with ∂^2 .

The vertex operators R^a can be considered in a similar way. As they are bosonic and act around a loop back to its initial position, without vanishing along the way, the part which acts on components of the loop may be considered to give a zero contribution. The R^a contributes its components to the loop regularisation, either through D^I or ∂^I . The latter cannot form $(\partial^I)^2$ with another ∂^I coming from vertex operators or propagators without any r transformed, though. Effectively, this brings with it that no $(\partial^I)^2$ can be formed, in total, and one R^a for each vertex must contribute D^I to the loop integration, for diagrams that need not be regularised with respect to r , which will be discussed further in section 4. For a summary of what all this means with respect to actual loop components, we now turn to the most simple example possible.

Illustrative example: the 4-point one-loop amplitude

The simplest case there is for a loop diagram is the one-loop amplitude which consists of four propagators and four 3-point vertices. In order to illustrate the properties presented above, we will have a closer look at the non-zero components of it.

It contains at most a power of $r^{4 \times 7} = r^{28}$ where no D is present and the presence of a D comes at the expense of one r , yielding that all terms with r^x , $x > 23$ vanish. Therefore, just as for bubbles and triangles, the conversion between r and D need not be considered and the maximal contribution of D^I to the loop regularisation is 9, the minimum required. This happens when one propagator contributes with $(D^I)^2$ through the presence of a term b_1 , compare subsection 2.3, and the rest of the vertices and propagators contribute with one D^I each through the presence of four R_1 and three $b_0 + b_1 + b_2$, the latter which need to be split between momenta (apart from b_2) so that they contain one D^I each. Moreover, due to the lack of the r^0 term in the antisymmetrisation of the split b operators, as mentioned right before this example, not two of the propagators can be represented by b_0 . Compare figure 3.

To conclude, it is appropriate to comment on the change of this once several loops are taken into account, as also is illustrated for a simple example in figure 3. It is

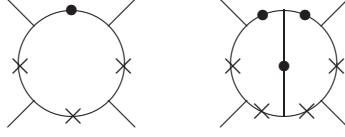


Figure 3: The distribution of terms on the one-loop diagram, as well as a possible distribution for the two-loop diagram. The filled dots represent b_1 terms with D^2 and the \times 's terms of b that only contribute with one D to the loop integration(s), just as each external vertex does. The internal vertices each contribute one D to both loops.

not possible to shift a b operator onto a propagator with different loop variables from the ones in the operator itself, but some relations can be deduced even so. Within each loop, the rules above apply. For example, for the two-loop diagram in figure 3, which does not need to be regularised with respect to r , at most three propagators can contribute D^2 to the loop integrations, one for each loop and one split between the loops, placed on the propagator they share. The corresponding is true for the vertex operators as well.

Furthermore, the property that no term with $(\partial^I)^2$ can be formed applies for all loop structures that have not been regularised with respect to r , even the ones with multiple loops. This however comes very close to having less to do with general properties than the minimal structure. For example, at least one D is required from each operator (acting into the loop) for the amplitude to be non-zero. This will in part be discussed a bit in the next subsection, and furthermore in the next section, on the UV properties of the amplitude diagrams, where we examine what is required of a loop structure in order for it to need to be regularised with respect to r .

Generally, it is difficult to discern properties of multi-loop amplitudes where r is regularised. The descriptions simply allow for a lot of freedom as to what the variables are and how they may act, which is in need of further examination.

3.4.3 The structure of loop diagrams

We now turn to a general inventory of the components of a loop structure with L loops, which e.g. will be of use in the next section when the question of when it is necessary to do a regularisation with respect to r is addressed. Temporarily disregarding the one-loop, we may observe that loop structures built from only 3-point vertices and without outer legs connected to them contain $3(L - 1)$ propagators and $2(L - 1)$ vertices, compare figure 4. To go from a structure with L loops to a structure with one loop more, a propagator must be added to the diagram, and its two ends connected to already existing propagators, that each become divided in two, yielding a total of three propagators and two vertices added to the diagram with L loops. The total number of propagators in a L -loop structure thus is

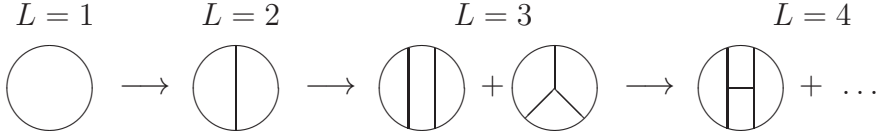


Figure 4: Illustration of how the number of propagators in a loop structure without external legs goes as $3L - 3$ for $L \geq 2$ when the 4-point vertex is absent, while the number of vertices goes as $2L - 2$. It also shows that the minimal amount of outer legs necessary in order to avoid bubbles and triangles for all diagrams at a certain loop order is 4 for $L \leq 2$, 3 for $L = 3$ and no more than 2 for $L \geq 4$, if there is no 4-point vertex present. Otherwise, the minimal j need to be increased by one outer leg, and $j = 2$ is not reached until $L = 5$.

$3(L - 1) + j$, with j the number of outer legs. This is at least four for $L \leq 2$, in order to avoid bubbles and triangles. For $L = 3$ at least 3 outer legs are necessary in order to get one non-zero amplitude, and for $L \geq 4$ there are amplitudes that are non-zero for the minimal requirement²¹ of $j = 2$. The total number of vertices is $2(L - 1) + j$.

The above is, of course, true only in the absence of the 4-point vertex, which however can show up only once in an amplitude, as described in subsection 2.3.3. When this special case occurs, the 4-point vertex can be regarded as having taken the place of a 4-point tree part as the one shown in figure 2. By this analogy, two 3-point vertices and one propagator are turned into the 4-point vertex, giving at hand the presence of one 4-point vertex, $3(L - 1) + j - 1$ propagators and $2(L - 2) + j$ 3-point vertices.

3.4.4 The importance of components outside a loop structure

A loop may appear alone, only accompanied by external fields, or exist as a part of a loop structure, i.e., several loops that are connected by propagators. Such a structure possibly has tree diagrams extending from it, resulting in external fields. No matter what the appearance, at least 4 external fields are necessary for an amplitude diagram to exist [12], and at least two legs need to be connected to the loop structure, as mentioned above.

The latter occurs since vacuum amplitudes (no outer legs connected to the loop structure) or amplitudes with no more than one external leg cannot occur [12]. Moreover, were a loop structure to be connected to one outer leg representing a tree diagram, the resulting expression would contain a total derivative (b) on the superfields, from the propagator connecting the tree part to the loop structure.

Consequently, at least two outer legs need to be attached to a loop cluster in order for the resulting expression to be non-zero. Taking the possibility of attaching

²¹The reason for this will be explained in the next subsection.

tree diagrams to the loop structure into consideration, it is also possible to see that two outer legs are enough for a non-zero result (for some diagrams). Four external fields can be present and the parts of the propagators that originate outside of the loop structure split up in a way as to not present a total derivative on the entire amplitude diagram.

With this, we conclude the section on the description of amplitude diagrams and move on to their UV properties. Note that there may be several more features of the amplitude diagrams, not described above. As previously mentioned though, the structure is complex and difficult to discern.

4 Ultraviolet behaviour of amplitude diagrams

In the presence of loops, integrations over the free momenta p^I are introduced. These integrations are performed up to some large momentum cut-off limit Λ , which typically threatens to make the expressions divergent. The infrared behaviour of this needs a detailed investigation, and will not be discussed here, but the examination of the ultraviolet behaviour is rather straightforward. The dependence on the momentum cut-off goes as:

$$\Lambda^{LD-2m+n} \quad (4.1)$$

Here, L is the number of loops, D the dimension (e.g. after a dimensional reduction), m the number of propagators that are part of the loops and n the number of loop momenta that can be constructed and paired up into $(\partial^I)^2$ in the loops. The theory is UV divergent if [12]

$$LD - 2m + n \geq 0 \quad (4.2)$$

As discernible above, the dimensional dependence enters through the integration over the loop momenta p^I , which there is D of in each of the L loops. The second term in the expression $(-2m)$ represents the UV contribution of loop momenta from the propagators:

$$\frac{1}{(p + p^I)^2} \sim \frac{1}{(p^I)^2} (1 + O(p/p^I)) , \quad p^I \gg p \quad (4.3)$$

Moreover, the last term of n must be even since parts that contain unpaired ∂^I 's give a zero contribution at the integration over dp^I .

An interesting question is whether or not maximal supergravity is finite in, say, four dimensions. In specific, what conditions does the framework of the formalism presented above set on the theory, in order for it to be finite? This question is what we address below, first by a look at what loop structures look like with respect to

their UV contribution, and later on by discerning what parts of them give at hand the worst dependence on the momentum cut-off. Lastly, we look at what this worst dependence looks like, for as it turns out, it is generally provided by the amplitude diagrams that need regularisation with respect to the number of r 's.

4.1 UV characteristics

The only part of an amplitude diagram which contributes to the UV behaviour, brought on by the loop regularisation, is the loop structure. The behaviour is described by eq. (4.1), where only one part is unknown, the term n describing the number of loop momenta that can be paired up into scalars in the loop structure. Furthermore, the number n is only unknown for amplitudes which are regularised with respect to r , a procedure we will refer to as just “regularisation” from here on.

Recall, from the previous section, that no quantity $(\partial^I)^2$ can be formed out of unregularised components. For diagrams that do not need to be regularised, we therefore can conclude that the worst UV dependence appears for diagrams only consisting of 3-point vertices, for which the number of outer legs are as few as possible, in the following way:

$$\Lambda^{L(D-6)+6-2j} \quad (4.4)$$

The reason for the restriction to a consideration of the minimal j is obvious: it is what gives the worst UV behaviour, which is what we are interested in. Moreover, the demand on D in order for this exponent not to be negative for any L or j is not more severe than $D < \frac{11}{2}$. This would arise from $L = 4$, $j = 2$, as the minimal j 's increase for low L 's, if indeed that amplitude does not need to be regularised. The question which remains to be answered is which conditions we have in general, as a function of L .

In order to determine the worst possible UV behaviour in general, we need to know when it is necessary to perform regularisation with respect to r , and how to interpret the components of the amplitude diagrams. This procedure might be anticipated to be tedious, but there is a convenient simplification for the case when the worst UV divergence is sought for, as is applicable here. It is possible to use the correspondence $r \leftrightarrow D$ and $D^2 \leftrightarrow \partial$ in the search for when and how to regularise, and ultimately in the search of n . As thus the problem is reduced to a matter of a counting of the power of r that is present, and an examination of whether or not it is allowed to be formed according to the amplitude characteristics. The problem separates into two parts. The first one refers to the actual power of r present, when the conditions for a requirement of regularisation are to be determined. The second refers to an effective power of r : $r^3/p^2 \sim r^{-1}$, which in the end gives at hand the

UV characteristics. Some properties are related though, which is why part of the discussion here will be performed simultaneously. We begin by having a look at the different parts of the amplitude diagrams, with respect to the above.

4.1.1 The insignificance of components outside the loop

Parts outside of the loop structure, e.g. external legs or parts of a tree structure connecting the loop structure to external legs, do not change the UV behaviour of the loop. The only way they could do so, by causing a higher power of r to be present and further regularisations to be necessary, provides only a superficial change, as noted in subsection 3.3.2. As such, we consider only the loop structure in order to determine the properties of it.

In supergravity, where vertex operators are present, this also means that operators that are associated with outer vertices can be assumed to act out of the loop as much as possible. Which leg each of them act on is arbitrary, yielding the same result, and thus the version where as many r 's as possible get outside of the loop, not to be included in the loop regularisation, reflects the real properties of the loop with the least intermediate examination. For example, both 3- and 4-point vertices (with 2 legs as part of the loop structure) can be taken to contribute with R^a only.

Furthermore, for regularised amplitudes, the number of outer legs is irrelevant. The conversion of a 3-point vertex into a 4-point one, with the new leg as an outer leg does not contribute anything new to the loop structure, as the T can be assumed to act out of the loop. Moreover, only one extra R^a is contributed to the loop by the addition of an outer vertex, and the additional b at most can contribute the equivalent of r^2 to the loop (since each loop only can have one outer propagator completely made out of loop variables), so that introduction of an outer vertex goes as $r^4/p^2 \sim p^0$. Consequently, for regularised amplitudes, the number of j is irrelevant.

4.1.2 The loop structure

Let us start by counting the maximal power of r that may be present in a loop structure. Beginning with the b operator, we may note that its maximal contribution corresponds to r^3 . Similarly, the $R^a \sim r^2$ and $T \sim r^3$. Consequently, the 3-point vertex is represented by r^4 and the 4-point vertex by r^7 . Furthermore, an amplitude diagram with only 3-point vertices contains $3(L - 1) + j$ propagators and $2(L - 1) + j$ vertices, compare subsection 3.4.3.

The presence of a 4-point vertex decreases the number of propagators by one, and the number of 3-point vertices by two, giving at hand that the maximal power of r present, for internal vertices, are *less* by a factor r^4 compared to a diagram of only 3-point vertices. For outer vertices, the difference is at most r^{-5} , and

the presence of the 4-point vertex can be recognised as insignificant due to the following reasons:

- For regularised amplitudes, the effective power of r is unaffected by the 4-point vertex. The presence of $1/p^2 \sim r^{-4}$ in the propagator, so the 4-point vertex and the 4-point tree amplitude built from 3-point vertices are indistinguishable in the power counting of r 's. Outer vertices do not need to be considered, as mentioned in the previous subsection.
- For low values of L , the presence of the 4-point vertex increases the minimal number of j to be attached by one. This is true for up to four loops, and we will see that higher loops require regularisation. However, when j is increased by one, a new set of vertex operators is added and the maximal power of r increased by r^7 . Effectively, the maximal power of r is thus increased by at least r^2 by the presence of the 4-point vertex, and it need not be considered when the question of when regularisation with respect to r is needed is addressed.

In short, the above states that only 3-point vertices need to be considered. For outer legs, the contribution to the maximal power of r is r^5 , and the effective contribution is r^0 . Apart from that, the relevant numbers of propagators and inner vertices are $3(L-1)$ and $2(L-1)$. The contributions with respect to the maximal number of r for each such component are r^3 and r^4 respectively.

As such, the maximal power of r present in a loop diagram is given by the term $9(L-1) + 8(L-1) + 5j$. In order for this to give at hand new results with respect to the $r \leftrightarrow D$ regularisation, the power must exceed that of $r^{23} D^{9L}$:

$$8L + 5j > 40 \tag{4.5}$$

That is, when the condition above is fulfilled, the $r \leftrightarrow D$ conversion must be taken into account, and eq. (4.4) no longer holds. Note that this always is true for $L > 3$, part of which was assumed above when the 4-point amplitude was neglected. Moreover, it is true for any $L > 1$ when the number of outer legs is not minimal.

4.1.3 Regularised amplitudes

In the search of the quantity n , the effective power of r present in a diagram is to be considered. The different parts are listed above, giving at hand r to the power of $9(L-1) + 8(L-1)$. Out of these, $23 + 9L$ are claimed by the loop regularisations and the r 's that need to be left untouched. The rest, $8L - 40$, may be converted into $4L - 20$ p 's. Furthermore, for the worst UV dependence, they are assumed

to pair up into $(p^I)^2$'s to as great an extent as possible, which all of them can do as the number is even, and no further restrictions on the amplitude diagrams have been discerned. Thus $n = 4L - 20$, giving at hand a maximal dependence on the cut-off Λ :

$$\begin{aligned}\Lambda^{LD-2m+n} &= [m = 3(L-1), \quad n = 4L - 20] = \\ &= \Lambda^{L(D-2)-14}, \quad 8L + 5j > 40\end{aligned}\tag{4.6}$$

Where this result is not valid, the dependence on Λ is governed by the unregularised result. Moreover, note that the regularised result for $L = 4, j = 2$ is the same as if it had not been regularised, which originates in the property that the superfluous r^2 cannot form a term ∂^2 .

4.2 The UV dependence

To conclude, the previous subsection states that the the maximal dependences on Λ and the subsequent requirements on D in order for the theory to be UV finite are:

$$\begin{array}{lll} \Lambda^{D-2j} & L = 1, 4 \leq j \leq 6 & : \quad D < 2j \\ \Lambda^{2D-14} & L = 2, j = 4 & : \quad D < 7 \\ \Lambda^{3D-18} & L = 3, j = 3 & : \quad D < 6 \\ \Lambda^{L(D-2)-14} & \text{otherwise} & : \quad D < 2 + \frac{14}{L}\end{array}\tag{4.7}$$

This fits exactly with previous results, for example the ones in ref. [18] for $L \leq 4$, and in refs. [11, 12] for all L 's, and points towards a possible divergence at $L = 7$.

5 Conclusions and outlook

The characteristics of the amplitude diagrams in maximal supergravity are intricate, and so far the description is not exhaustive. There may yet be further properties to be discovered, that would affect the convergence of the diagrams in the UV regime. So far though, the maximally supersymmetric description with manifest supersymmetry through the use of pure spinors, has not yielded any other results than the first quantised version, based on $D = 10$ pure spinors, presented in ref. [12]. The predictions contain the possible divergence of $D = 4, \mathcal{N} = 8$ supergravity at seven loops.

This is to be compared to maximal SYM, which is known to be perturbatively finite in four dimensions, a fact that is predicted by a similar examination as above, but for SYM. The principles are the same though the degrees of freedom differ²²

²²There are 11 degrees of freedoms for pure spinors and otherwise 16 (compare 23 and 32) so that 5 D 's are required for a loop integration.

and no vertex operators are present. The result for the unregularised case is the same as in supergravity, followed by $D < 4 + \frac{6}{L}$ for $L \geq 2$.

Of course, this does not prevent that other characteristics, not yet observed, might come into play and soften the UV divergence. The main ingredient that may impose further restrictions on the behaviour of amplitudes is U-duality. However, it is not known how to incorporate U-duality in a formulation with pure spinors, and a classical formulation possessing both manifest U-duality and supersymmetry is still lacking. U-duality, in $D = 4$ manifested as a continuous $E_{7(7)}$ symmetry of perturbative amplitudes, has been shown to exclude 4-point counterterms up to six loops, while it seems to leave room for a possible divergence at seven loops [13]. It is striking that supersymmetry and U-duality seem to yield the same non-renormalisation properties, and it would indeed be interesting if they could be combined.

Essentially, the more detailed amplitude calculations of ref. [18] have not yet surpassed the boundary where the full properties of the loop regularisation come into play, at five loops. Once that is the case, it might be possible to predict the true UV divergence with more reliability.

Acknowledgments

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A Spinor and pure spinor identities in $D = 11$

The spinors of supergravity in $D = 11$ are symplectic. A spinor index is raised by the antisymmetric tensor $\varepsilon^{\alpha\beta}$, the presence of which is usually left implicit. Moreover, note the convention which we use for the antisymmetrisation of indices:

$$(\gamma_{ab})_{\alpha\beta} = \frac{1}{2}[(\gamma_a\gamma_b)_{\alpha\beta} - (\gamma_b\gamma_a)_{\alpha\beta}] \quad (\text{A.1})$$

Fierz rearrangements are always made between spinors at the right and left of two spinor products, and the general Fierz identity is:

$$(AB)(CD) = \sum_{p=0}^5 \frac{1}{32p!} (C\gamma^{a_1\dots a_p}B)(A\gamma_{a_p\dots a_1}D) \quad (\text{A.2})$$

In the above, the spinors have been assumed to be bosonic, but the relation holds for all operators, provided that the appropriate sign (for the statistics of the operators) is added. In specific, for bilinears in a pure spinor λ this reduces to:

$$(A\lambda)(\lambda B) = -\frac{1}{64}(\lambda\gamma^{ab}\lambda)(A\gamma_{ab}B) + \frac{1}{3840}(\lambda\gamma^{abcde}\lambda)(A\gamma_{abcde}B) \quad (\text{A.3})$$

This gives at hand some useful identities for pure spinors, among which are:

$$\begin{aligned}
(\gamma_j \lambda)_\alpha (\lambda \gamma^{ij} \lambda) &= 0 \\
(\gamma_i \lambda)_\alpha (\lambda \gamma^{abcdi} \lambda) &= 6(\gamma^{[ab} \lambda)_\alpha (\lambda \gamma^{cd]} \lambda) \\
(\gamma_{ij} \lambda)_\alpha (\lambda \gamma^{abcij} \lambda) &= -18(\gamma^{[a} \lambda)_\alpha (\lambda \gamma^{bc]} \lambda) \\
(\gamma_{ijk} \lambda)_\alpha (\lambda \gamma^{abijk} \lambda) &= -42\lambda_\alpha (\lambda \gamma^{ab} \lambda) \\
(\gamma_{ij} \lambda)_\alpha (\lambda \gamma^{abcdij} \lambda) &= -24(\gamma^{[ab} \lambda)_\alpha (\lambda \gamma^{cd]} \lambda) \\
(\gamma_i \lambda)_\alpha (\lambda \gamma^{abcdei} \lambda) &= \lambda_\alpha (\lambda \gamma^{abcde} \lambda) - 10(\gamma^{[abc} \lambda)_\alpha (\lambda \gamma^{de]} \lambda)
\end{aligned} \tag{A.4}$$

Furthermore, it is possible to act on the identity $(\bar{\lambda} \gamma^{[ij} \bar{\lambda})(\bar{\lambda} \gamma^{kl]} \bar{\lambda}) = 0$ with $\bar{\partial}$ and make use of the constraint on the spinor r , $(\bar{\lambda} \gamma^a r) = 0$, in order to derive:

$$(\bar{\lambda} \gamma^{[ij} \bar{\lambda})(\bar{\lambda} \gamma^{kl]} r) = 0 \tag{A.5}$$

Other useful relations, especially related to calculations with structures as in eq. (2.31), are:

$$\begin{aligned}
(\bar{\lambda} \gamma^i_k \bar{\lambda})(\bar{\lambda} \gamma^{jk} r) &= (\bar{\lambda} \gamma^{ij} \bar{\lambda})(\bar{\lambda} r) \\
(\bar{\lambda} \gamma^i_k r)(\bar{\lambda} \gamma^{jk} r) &= (\bar{\lambda} \gamma^{ij} r)(\bar{\lambda} r) + \frac{1}{2}(\bar{\lambda} \gamma^{ij} \bar{\lambda})(rr) \\
(\bar{\lambda} \gamma^i_k \bar{\lambda})(\bar{\lambda} \gamma^k_l r)(\bar{\lambda} \gamma^{lj} r) &= 0 \\
(\bar{\lambda} \gamma^i_k r)(\bar{\lambda} \gamma^k_l r)(\bar{\lambda} \gamma^{lj} r) &= 0
\end{aligned} \tag{A.6}$$

Further relations can of course be deduced, but the above constitute the main ones used in this paper.

B The zero-mode cohomology of ψ

The condition $(\lambda D)\psi = 0$, the equations of motion for a free field in the minimal formalism, allows for certain non-zero components of ψ , independent of x , a set which is termed to be the zero-mode cohomology of ψ . In specific, the variables (λ, θ) must be put together so that they form special structures, defined up to some general fields with certain properties. This is what is listed in table 1.

In specific, what the table shows is the field components that are independent of (λ, θ) , listed with respect to their irreducible representations at the positions which give at hand their ghost numbers and dimensions²³. That is, the element in the upper left corner represents a field in the irreducible representation of (00000) with ghost number 3 and dimension -3 , which represents cohomology in ψ without being attached to (λ, θ) . One step to the right represents allowing for one pure spinor in the superfield, decreasing the ghost number and the dimension of a field component to 2 and $-5/2$ respectively, though that spot is empty, represented by

²³Recall that λ is of ghost number 1, whereas θ is of ghost number 0. Both are spinors and as such of dimension $-1/2$. The superfield, on the other hand, has ghost number 3 and dimension -3 .

a dot. In order to allow for zero-mode cohomology, at least one θ must be added. This brings us one step further down and allows for a field in the representation of (10000), connected to $\lambda\gamma^a\theta$, and so on.

Even though the zero-mode cohomology, the calculation of which is a purely algebraic problem, represents a huge simplification compared to the actual cohomology, it has a very concrete physical interpretation. It is clear that the reintroduction of the term in (λD) which contains a derivative with respect to x will impose a further restriction compared to the zero-mode cohomology. The full cohomology will be represented by elements in the zero-mode cohomology restricted by some differential equations. These equations will be the equations of motion of the physical *fields* (including ghosts and antifields), which in turn are the elements of the zero-mode cohomology. In order for such equations of motion to contain non-trivial information, they must in turn be represented by elements in the zero-mode cohomology, but (since λD has ghost number 1) at the next higher power of λ . It is for this reason that both component fields and antifields (which of course represent equations of motion) appear in the cohomology, and the models inherently become meaningful only in a field–antifield, i.e., Batalin–Vilkovisky framework.

C The projector on the pure spinor cotangent space

What may be used in order to ensure the properties of the projection $P^\alpha{}_\beta$ in eq. (3.3), described in subsection 3.2, is the Fierz identity $(\gamma_b\lambda)_\alpha(\lambda\gamma^{ab}\lambda) = 0$. Therefore, we write an ansatz for P_\perp :

$$P^\alpha{}_\beta = (\gamma_a\bar{f})^\alpha R^a{}_b (\lambda\gamma^b)_\beta \equiv \Pi^\alpha{}_\beta[R] \quad (\text{C.1})$$

Here, R is some, yet undetermined, matrix, \bar{f} is an arbitrary non-vanishing pure reference spinor, and the last equality just represents a convenient notation. It is not immediately obvious how to best make an ansatz for R , since there is a number of independent expressions containing an equal number of λ 's and \bar{f} 's. In addition to the unit matrix, there are two antisymmetric matrix structures and two symmetric. They may be taken as:

$$\begin{aligned} M_{ab} &\equiv (\lambda\gamma_{ab}\bar{f}) & S_{ab}^{(2)} &\equiv (\lambda\gamma_{(a}{}^i\lambda)(\bar{f}\gamma_{b)i}\bar{f}) \\ A_{ab}^{(2)} &\equiv (\lambda\gamma_{[a}{}^i\lambda)(\bar{f}\gamma_{b]i}\bar{f}) & S_{ab}^{(5)} &\equiv (\lambda\gamma_{(a}{}^{ijkl}\lambda)(\bar{f}\gamma_{b)ijkl}\bar{f}) \end{aligned} \quad (\text{C.2})$$

However, since the matrices appear sandwiched between $\gamma_a\bar{f}$ and $\lambda\gamma^b$, $A^{(2)}$ and $S^{(2)}$ are equivalent: $\Pi[A^{(2)}] = \Pi[S^{(2)}]$. Alternatively, one of the antisymmetric expressions may be replaced by $A_{ab}^{(5)} \equiv (\lambda\gamma_{[a}{}^{ijkl}\lambda)(\bar{f}\gamma_{b]ijkl}\bar{f})$, thanks to the Fierz identity $\alpha M_{ab} = \frac{1}{16}A_{ab}^{(2)} - \frac{1}{384}A_{ab}^{(5)}$. There are two scalar invariants, $\alpha = (\lambda\bar{f})$ and

dim \ gh#	3	2	1	0	-1	-2	-3	-4	-5
-3	(00000)								
$-\frac{5}{2}$	•	•							
-2	•	(10000)	•						
$-\frac{3}{2}$	•	•	•	•					
-1	•	•	(01000) (10000)	•	•				
$-\frac{1}{2}$	•	•	(00001)	•	•	•			
0	•	•	•	(00000) (00100) (20000)	•	•	•		
$\frac{1}{2}$	•	•	•	(00001) (10001)	•	•	•	•	
1	•	•	•	•	•	•	•	•	•
$\frac{3}{2}$	•	•	•	•	(00001) (10001)	•	•	•	•
2	•	•	•	•	(00000) (00100) (20000)	•	•	•	•
$\frac{5}{2}$	•	•	•	•	•	(00001)	•	•	•
3	•	•	•	•	•	(01000) (10000)	•	•	•
$\frac{7}{2}$	•	•	•	•	•	•	•	•	•
4	•	•	•	•	•	•	(10000)	•	•
$\frac{9}{2}$	•	•	•	•	•	•	•	•	•
5	•	•	•	•	•	•	•	(00000)	•

Table 1: The zero-mode cohomology in ψ . The horizontal direction represents the expansion of the superfield in terms of λ whereas the corresponding for the vertical (in each row) is θ (downward). The irreducible representations of the component fields are listed at the positions which describe their ghost numbers and dimensions.

$\beta = (\lambda\gamma_{ab}\lambda)(\bar{f}\gamma^{ab}\bar{f})$, and the ansatz for the matrix will consist of functions of these invariants multiplying the matrices. More precisely, we will have functions of the dimensionless invariant $x = \frac{\beta}{8\alpha}$, and everywhere the appropriate power of α to make all terms dimensionless. For simplicity we will temporarily set $\alpha = 1$ below.

In doing this, it is easier to work with a basis for the matrices consisting of powers of $M^a_b = (\lambda\gamma^a_b\bar{f})$. This is because the multiplication rule is

$$\Pi[R]\Pi[S] = \Pi[R(\mathbb{I} + M)S] \quad (\text{C.3})$$

which follows immediately from the definition of $\Pi[R]$. It therefore follows that $\Pi[(\mathbb{I} + M)^*]$ is a projection matrix, where the operation R^* denotes a “weak inverse” in the sense that $\Pi[R^*R] = \Pi[\mathbb{I}]$. This far, the construction is completely general and independent of dimension. The specific information will turn up in an effective Cayley–Hamilton equation for the matrix M , which will be used in the form

$$\Pi[p(M)] = 0 \quad (\text{C.4})$$

where p is some polynomial. From the considerations above, the degree of p must be²⁴ 4. Multiplication by M is easy for M , and also for $A^{(2)}$ and $S^{(2)}$, obeying $MA^{(2)} = S^{(2)}$, $MS^{(2)} = A^{(2)}$. A calculation with some Fierz rearrangements leads to

$$M^3 - (1 - x)M = \frac{1}{2}A^{(2)} \quad (\text{C.5})$$

$$M^4 - (1 - x)M^2 = \frac{1}{2}S^{(2)} \quad (\text{C.6})$$

and subtraction of these two equations leads to an equivalence of the form in eq. (C.4) with

$$p(M) = M^4 - M^3 - (1 - x)M^2 + (1 - x)M \quad (\text{C.7})$$

Note that the Cayley–Hamilton relation $\Pi[p(M)] = 0$ is stronger than, but consistent with, the one for the matrix M itself:

$$M^5 - (2 - x)M^3 + (1 - x)M = 0 \quad (\text{C.8})$$

The projection matrix can now be calculated using this stronger Cayley–Hamilton relation, and we find:

$$\Pi[(\mathbb{I} + M)^*] = \Pi[\mathbb{I} - \frac{1+x}{2x}M + \frac{1}{x}M^2 - \frac{1}{2x}M^3] \quad (\text{C.9})$$

²⁴Note that in $D = 10$, the degree is 1, and $\Pi[M - \mathbb{I}] = 0$.

This can of course be translated to another basis (the basis $\{\mathbb{I}, M, M^2, M^3\}$ has the disadvantage of containing structures with unnecessarily high powers of λ and \bar{f}), see below.

We can verify that this projection indeed is on the complement to the tangent space of pure spinor space by calculating its rank. Using $\text{tr } \Pi[R] = \text{tr}(\mathbb{I} + M)R$, and

$$\text{tr } \mathbb{I} = 11 \quad (\text{C.10})$$

$$\text{tr } M^2 = 2(5 - 3x) \quad (\text{C.11})$$

$$\text{tr } M^4 = (1 - x) \text{tr } M^2 + 4x = 2(5 - 6x + 3x^2) \quad (\text{C.12})$$

we arrive at $\text{tr } \Pi[(\mathbb{I} + M)^*] = 9$, so this is indeed the correct projection.

A gauge invariant derivative can be formed as:

$$W_\alpha = (P^t w)_\alpha \quad (\text{C.13})$$

Its invariance, when $P^\alpha_\beta = \delta^\alpha_\beta - \Pi^\alpha_\beta[R]$, means that:

$$(\lambda \gamma^a)_\alpha - ((\mathbb{I} + M)R)^a_b (\lambda \gamma^b)_\alpha = 0 \quad (\text{C.14})$$

This invariance follows directly from the general relations above (in any dimension), from the observation that the multiplication with $\lambda \gamma^b$ on the right is sufficient to imply the stronger Cayley–Hamilton relation. This projected derivative can be seen as the covariant derivative (with $\bar{f} \rightarrow \bar{\lambda}$) corresponding to the metric obtained as the pullback from flat 32_C-dimensional spinor space.

A nicer basis might be the one based on the matrices $\mathbb{I}, M, A^{(2)} \approx S^{(2)}$ and $S^{(5)} - A^{(5)}$. The last matrix is chosen since its contribution can be rewritten using

$$\Pi[S^{(5)} - A^{(5)}] = 36(\gamma_{[ij}\bar{f})^\alpha(\bar{f}\gamma_{kl]}\bar{f})(\lambda\gamma^{ij}\lambda)(\lambda\gamma^{kl})_\beta \quad (\text{C.15})$$

The expression for the projector to tangent space then is

$$\begin{aligned} P^\alpha_\beta &= \delta^\alpha_\beta - \frac{1}{4\alpha}(\gamma_a \bar{f})^\alpha(\lambda \gamma^a)_\beta - \frac{1}{2\alpha\beta}(\gamma_a \bar{f})^\alpha(\lambda \gamma^{ai}\lambda)(\bar{f}\gamma_{bi}\bar{f})(\lambda \gamma^b)_\beta + \\ &+ \frac{3}{4\alpha\beta}(\gamma_{[ij}\bar{f})^\alpha(\bar{f}\gamma_{kl]}\bar{f})(\lambda \gamma^{ij}\lambda)(\lambda \gamma^{kl})_\beta \end{aligned} \quad (\text{C.16})$$

since the coefficient of $\Pi[M]$ vanishes. Here, the explicit powers of α have been reinserted. From this form we can more easily verify that no terms blow up when one approaches the codimension 7 subspace with $(\lambda \gamma_{ab}\lambda) = 0$.

A “translated” λ is obtained by exponentiating:

$$\lambda' = e^{(\varepsilon W)} \cdot \lambda^\alpha = \lambda^\alpha + \varepsilon^\alpha - \frac{1}{2}(\gamma_a \bar{f})^\alpha R^a_b(\lambda + \varepsilon, \bar{f})((\lambda + \varepsilon)\gamma^b(\lambda + \varepsilon)) \quad (\text{C.17})$$

This shows that a regulator whose action on the bosonic variables is given by eq. (3.2) provides enough smoothing around $\eta = 0$ to render bosonic integration finite, as described in subsection 3.2.

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